11/9/00 Solutions to practice problems

(1) (a) i^3-i^2+i-1

$= i^2 \cdot i-(-1)+i-1$

$= \cancel{-i}\ \cancel{+1}\ \cancel{+i}\ \cancel{-1} = \underline{\underline{0}}+0i$

(b) $\dfrac{5+3i}{7-8i}$

$= \dfrac{5+3i}{7-8i} \times \dfrac{7+8i}{7+8i}$

$= \dfrac{5\cdot 7+40i+21i-24}{(7)^2-(8i)^2}$

$= \dfrac{35+61i-24}{49+64} = \dfrac{35-24}{113}+\dfrac{61i}{113}$

$= \underline{\underline{\dfrac{11}{113}+\dfrac{61i}{113}}}$

(c) $7e^{i\pi/6}+3e^{-i\pi/6}$

$= 7\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]+3\left[\cos\left(\pi/6\right)-i\sin\left(\frac{\pi}{6}\right)\right]$

$= 7\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)+3\left(\frac{\sqrt{3}}{2}-i\,\frac{1}{2}\right) = \underline{\underline{5\sqrt{3}-i}}$

(2) $\sum_{i=0}^{17} e^{i\pi n/6}$

$$= \frac{(1 - e^{i\pi/6 \cdot 18})}{1 - e^{i\pi/6}}$$

$$= \frac{1 - e^{i3\pi}}{1 - e^{i\pi/6}}$$

$$= \frac{2}{1 - \cos(\pi/6) - i\sin(\pi/6)} = \frac{2}{\left(1 - \frac{\sqrt{3}}{2}\right) - i\frac{1}{2}}$$

$$= \frac{2}{\left(\frac{2-\sqrt{3}}{2}\right) - i/2}$$

$$= \frac{4}{\underline{\underline{(2-\sqrt{3}) - i}}}$$

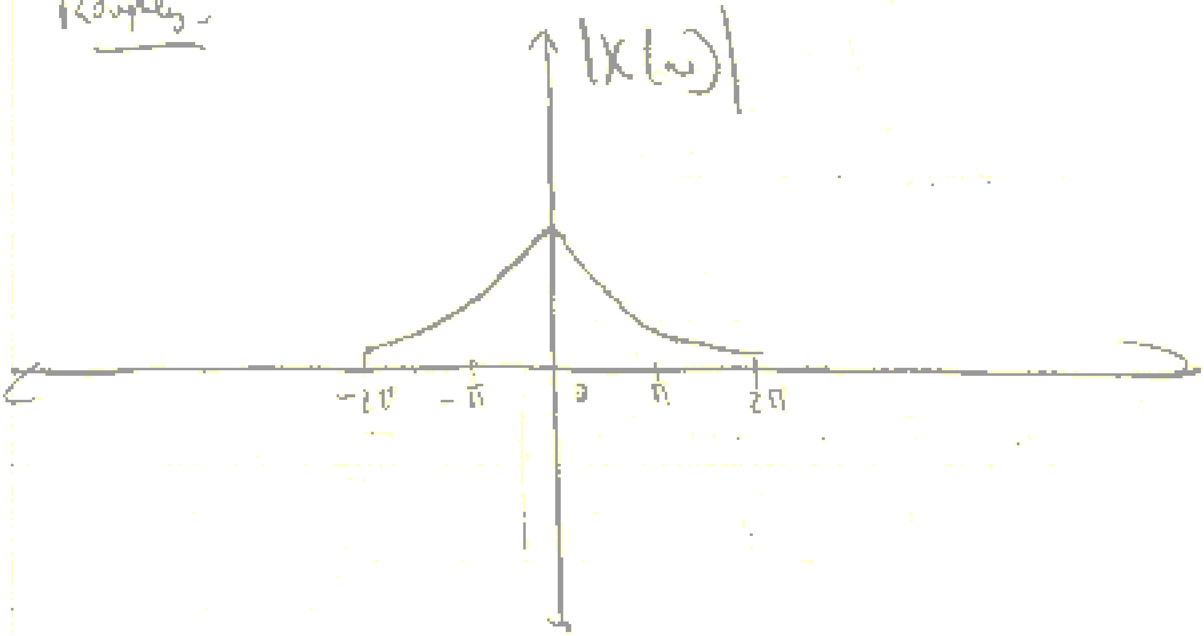
(3) $X(\omega) = \frac{1}{1 + i\omega}$

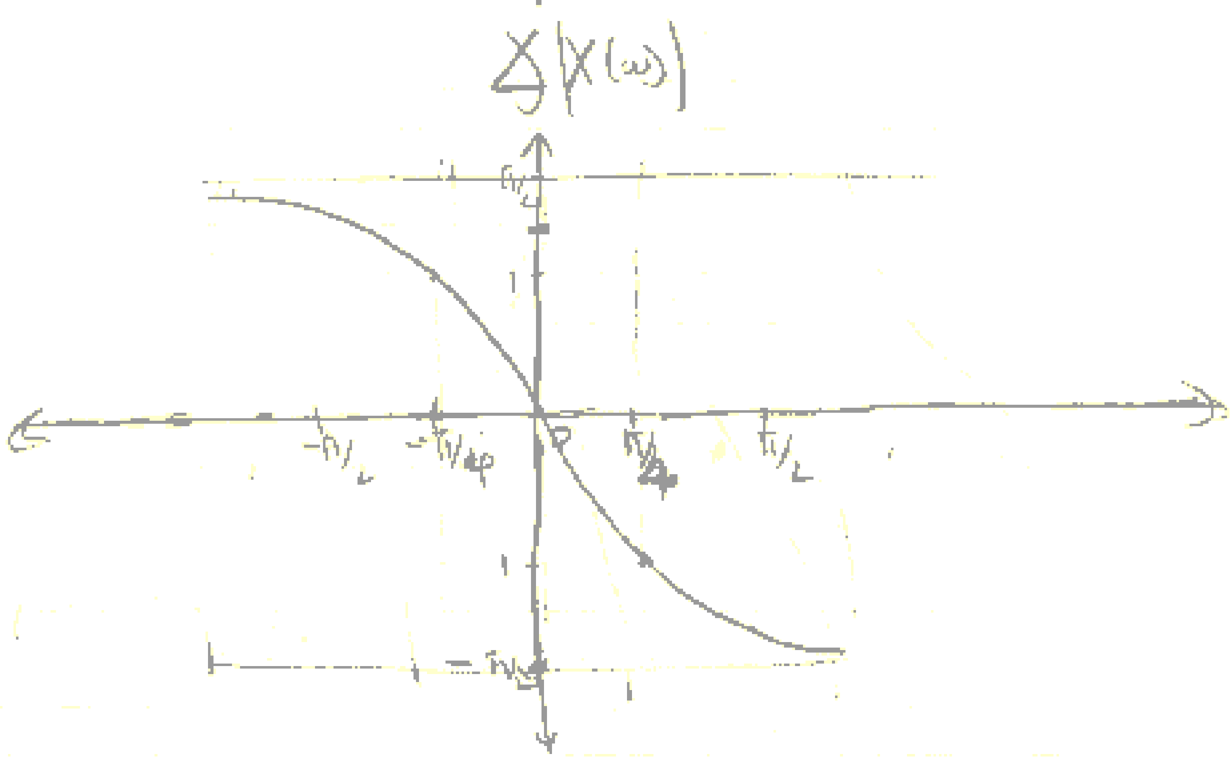
$|X(\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$ $\quad\quad \measuredangle X(\omega) = -\tan^{-1}(\omega)$

$\therefore \omega = -2\pi, 2\pi, \; |X(\omega)| = \dfrac{1}{\sqrt{1+4\pi^2}} \approx \underline{\underline{0.157}}$

$\omega = 0, \; |X(\omega)| = 1$

<u>Roughly:</u>





(4) Step (1): $P(1) \Rightarrow$ L.H.S $= re^{i\theta} = r[\cos\theta + i\sin\theta]$ [Euler's formula]

$= r^1[\cos\theta + i\sin\theta]$

$=$ R.H.S

$\Rightarrow$ P(1) is true

Step (2): Assume P(n) is true for some n = k i.e.

$P(k) = r^k = (re^{i\theta})^k = r^k[\cos k\theta + i\sin k\theta]$

Step (3): Prove P(k+1) is true:

L.H.S $= r^{k+1}$

$= (re^{i\theta})^{k+1}$

$= r^{k+1} e^{i\theta(k+1)}$

$= r^k . e^{i\theta k} . re^{i\theta}$

$= r^k[\cos k\theta + i\sin k\theta] . re^{i\theta}$ [By assumption]

$= r^{k+1}\left[(\cos k\theta + i\sin k\theta)\right](\cos\theta + i\sin\theta]$

[Euler's formula]

$$= r^{k+1}\left[\left(\cos k\theta \, i\sin\theta + \cos k\theta \cos\theta\right) + \left(i \sin k\theta \cos\theta - \sin k\theta \sin\theta\right)\right]$$

$$= r^{k+1}\left[\left(\cos k\theta \cos\theta - \sin k\theta \sin\theta\right) + i\left[\cos k\theta \sin\theta + \sin k\theta \cos\theta\right]\right]$$

$$= r^{k+1}\left[\cos (k+1)\theta + i\left[\sin (k+1)\theta\right]\right]$$

$$= R.H.S$$

$\Rightarrow$ $P(k+1)$ is true

$\rightarrow$ $P(n)$ is true by the principle of math. induction

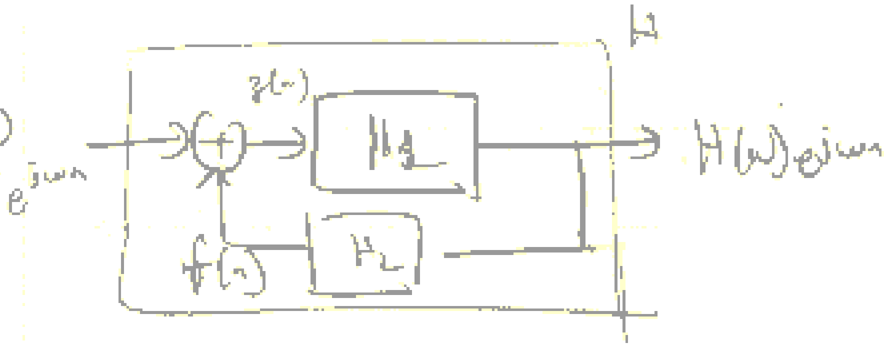
(2) a)

$$y(n) = \left[e^{j\omega n} + H(\omega)e^{j\omega n}\right]$$

$$\therefore H_1(\omega)\cancel{e^{j\omega n}} + H_1(\omega)H(\omega)\cancel{e^{j\omega n}} = H(\omega)\cancel{e^{j\omega n}}$$

$$\rightarrow \boxed{\frac{H_1(\omega)}{1 - H_1(\omega)} = H(\omega)}$$

(b)



$$\therefore \; f(n) = H_2(\omega) \cdot H(\omega) e^{j\omega n}$$

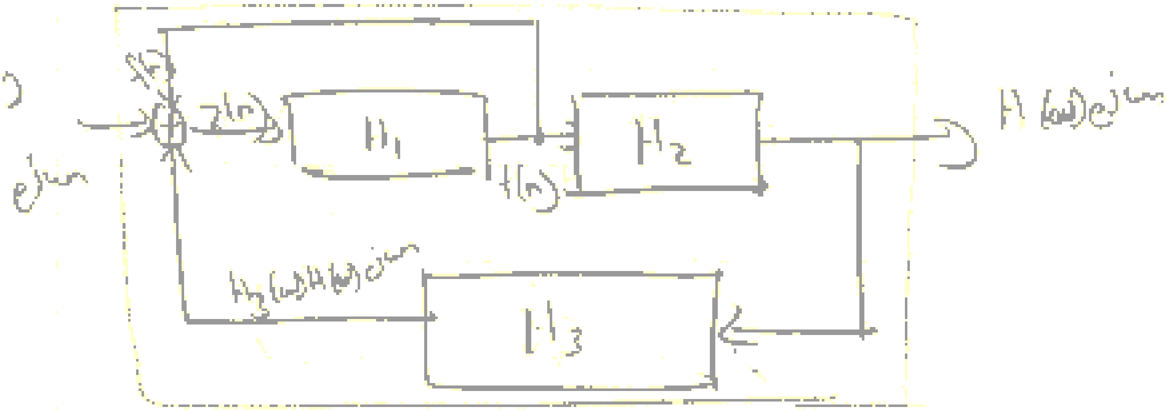
$$\therefore \; z(n) = e^{j\omega n} + H_2(\omega) H(\omega) e^{j\omega n}$$

$$\therefore \; H_1(\omega) z(n) = H(\omega) e^{j\omega n}$$

$$\Rightarrow H_1(\omega) \left[\cancel{e^{j\omega n}} + H_2(\omega) H(\omega) \cancel{e^{j\omega n}} \right] = H(\omega) \cancel{e^{j\omega n}}$$

$$\Rightarrow \boxed{\frac{H_1(\omega)}{1 - H_1(\omega) H_2(\omega)} = H(\omega)}$$

(c)



Doing this problem ad hoc is hard:

Now, $f(n) + e^{j\omega n} + H_3(\omega) H(\omega) e^{j\omega n} = z(n)$

But, $f(n) = H_1(\omega) z(n)$

$\Rightarrow H_1(\omega) z(n) + e^{j\omega n} + H_3(\omega) H(\omega) e^{j\omega n} = z(n)$

[Substitute for f(n)]

$$\Rightarrow z(n) = \frac{e^{j\omega n} + H_3(\omega) H(\omega) e^{j\omega n}}{1 - H_1(\omega)}$$

Hence, $f(n) = H_1(\omega) \left[\frac{e^{j\omega n} + H_3(\omega) H(\omega) e^{j\omega n}}{1 - H_1(\omega)} \right]$ — (2)

Thus, $H_2(\omega) . f(n) = H(\omega) e^{j\omega n}$ — (1)

$$\Rightarrow H_2(\omega) \left\{ H_1(\omega) \left[\frac{\cancel{e^{j\omega n}} + H_3(\omega) H(\omega) \cancel{e^{j\omega n}}}{1 - H_1(\omega)} \right] \right\}$$

[Substituting (2) in (1)]

$$= H(\omega) \cancel{e^{j\omega n}}$$

$$\Rightarrow \frac{H_2(\omega) H_1(\omega)}{1 - H_1(\omega)} + \frac{H_1(\omega) H_2(\omega) H_3(\omega) H(\omega)}{1 - H_1(\omega)}$$

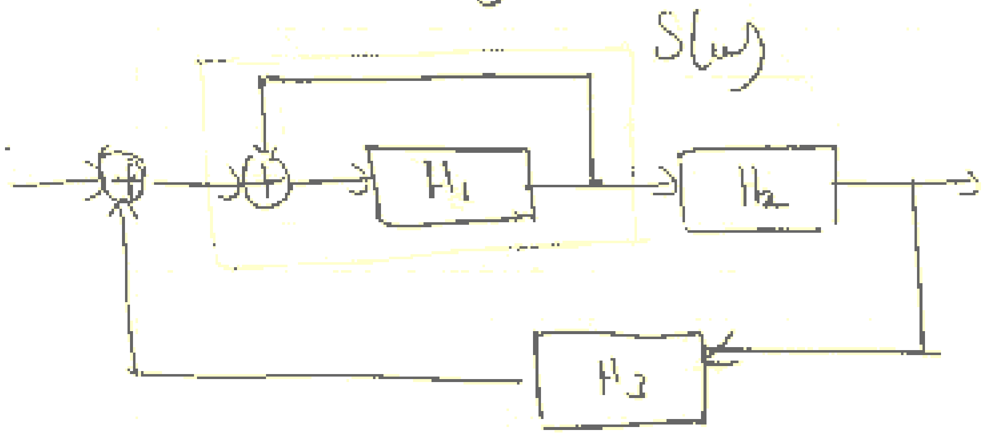
$$= H(\omega)$$

$$\Rightarrow H_2(\omega) H_1(\omega) + H_1(\omega) H_2(\omega) H_3(\omega) H(\omega) = H(\omega)\left[1 - H_1(\omega)\right]$$

$$\Rightarrow H_1(\omega) H_2(\omega) = H(\omega) - H_1(\omega) H(\omega) - H_1(\omega) H_2(\omega) H_3(\omega) H(\omega)$$

$$\Rightarrow \boxed{H(\omega) = \frac{H_1(\omega) H_2(\omega)}{1 - H_1(\omega) - H_1(\omega) H_2(\omega) H_3(\omega)}}$$

However, if you use the hint:

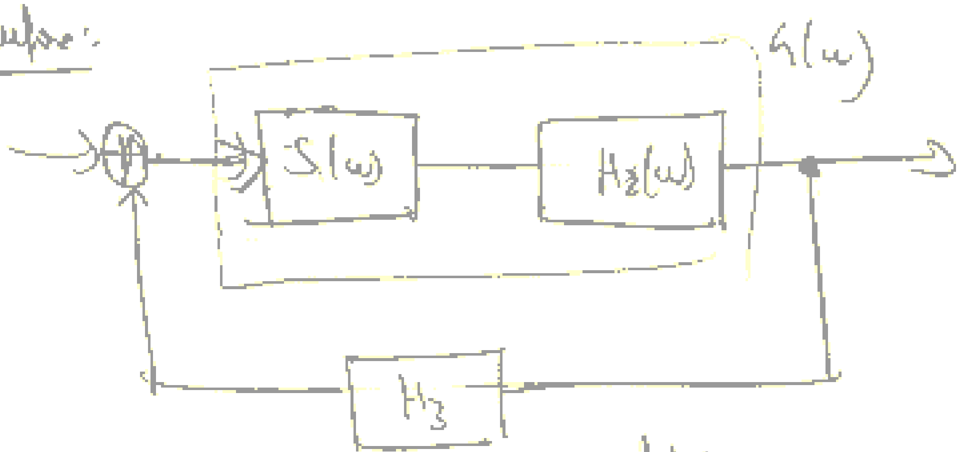


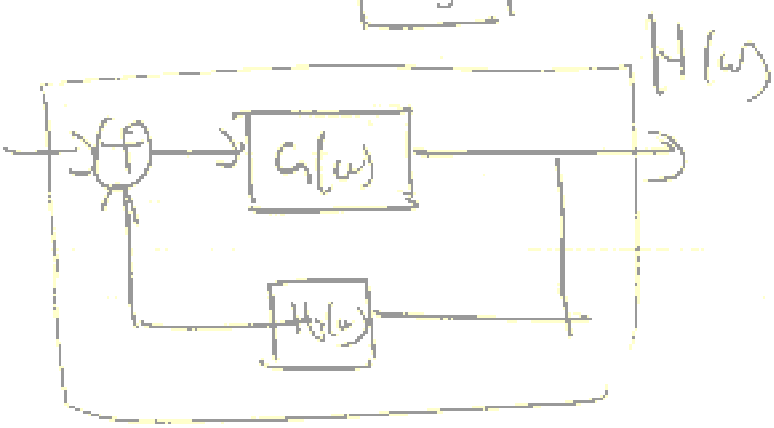
You know: $S(\omega) = \dfrac{H_1(\omega)}{1 - H_1(\omega)}$

and:

$$G(\omega) = S(\omega)\, H_2(\omega)$$

Therefore:





$$H(\omega) = \frac{G(\omega)}{1 - G(\omega) H_3(\omega)} = \frac{\dfrac{H_1(\omega) H_2(\omega)}{1 - H_1(\omega)}}{1 - \dfrac{H_1(\omega) H_2(\omega) H_3(\omega)}{1 - H_1(\omega)}}$$

$$\Rightarrow \boxed{H(\omega) = \frac{H_1(\omega) H_2(\omega)}{1 - H_1(\omega) - H_1(\omega) H_2(\omega) H_3(\omega)}}$$

Practice problem set # 2

(2) (i) Now $i(t) = C\dot{y}(t)$

$\therefore$ $x(t) = RC\dot{y}(t) + y(t)$

$$\Rightarrow \boxed{RC\dot{y}(t) + y(t) = x(t)}$$

(ii) Using the given definitions:

$$\boxed{RC\left[j\omega Y(\omega)\right] + Y(\omega) = X(\omega)} \quad \text{— (1)}$$

(iii) Solving for $\frac{Y(\omega)}{X(\omega)}$ in (1)

$$Y(\omega)\left[RC\,j\omega + 1\right] = X(\omega)$$

$$\Rightarrow \boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC}} \quad \text{— (2)}$$

(iv) Use the following commands:

```
>> w=[0:2*pi*1:10000];
>> H= 1./(1+(j*w).*/(2*pi*1000));
>> plot (w, abs(H))
```

>> plot(w, angle(H))

The plots are attached

(V) Ux: >> semilogx(w, ~~abs(H)~~ 20* log(abs(H)))

The plots are also attached.

(VI) from the plots: $|H(0)| \approx 1$ [>> abs(H(1)) % matlab array indices at 1
ans = 1]

$|H(2\pi 1000)| \approx \frac{1}{\sqrt{2}}$ [Check it by substituting $w = 2\pi 1000$ in (2)]

$|H(2\pi 10000)| \approx 0.09$ [check using (2)]

(VII) $y(t) = |H(2\pi 1000)| \sin(2\pi 1000t + \angle H(2\pi 1000))$

$$\boxed{y(t) = \frac{1}{\sqrt{2}} \sin(2\pi 1000t - \pi/4)}$$

(VIII) $$\boxed{y(t) = \frac{1}{\sqrt{2}} \sin(2\pi 1000t)}$$

(IX) $y(t) = |H(2\pi 10{,}000)| \sin(2\pi 10{,}000t + \angle H(2\pi 10{,}000))$

$\approx 0.099 \sin(2\pi 10{,}000t + (-1.47))$

Signal is attenuated, LOW-PASS FILTER

(2) (i) Now, $V_R(t) + V_L(t) + v(t) = x(t)$ [Kirchoff's voltage law]

$$\Rightarrow iR + L\frac{di}{dt} + v(t) = x(t)$$ [Ohm's law, $V = iR$]

But, $i = C\frac{dv}{dt}$

$$\therefore RC\frac{dv}{dt} + LC\frac{d}{dt}\left(\frac{dv}{dt}\right) + v(t) = x(t)$$

$$\Rightarrow \boxed{LC\frac{d^2y}{dt^2} + RC\frac{dy}{dt} + y(t) = x(t)}$$ [$v(t) = y(t)$, from the figure]

(ii) Using the same techniques from problem (1)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

and:

$$-LC\omega^2 Y(\omega) + RCj\omega Y(\omega) + Y(\omega) = X(\omega)$$

$$\Rightarrow \boxed{H(\omega) = \frac{1}{-LC\omega^2 + RCj\omega + 1}}$$

(iii) The matlab plots are attached

$|H(\omega)|$ decreases faster

& $\angle H(\omega)$ goes to $-\pi/2$ as $\omega \to \infty$ faster.

(iv) The drop is 40 dB/ decade

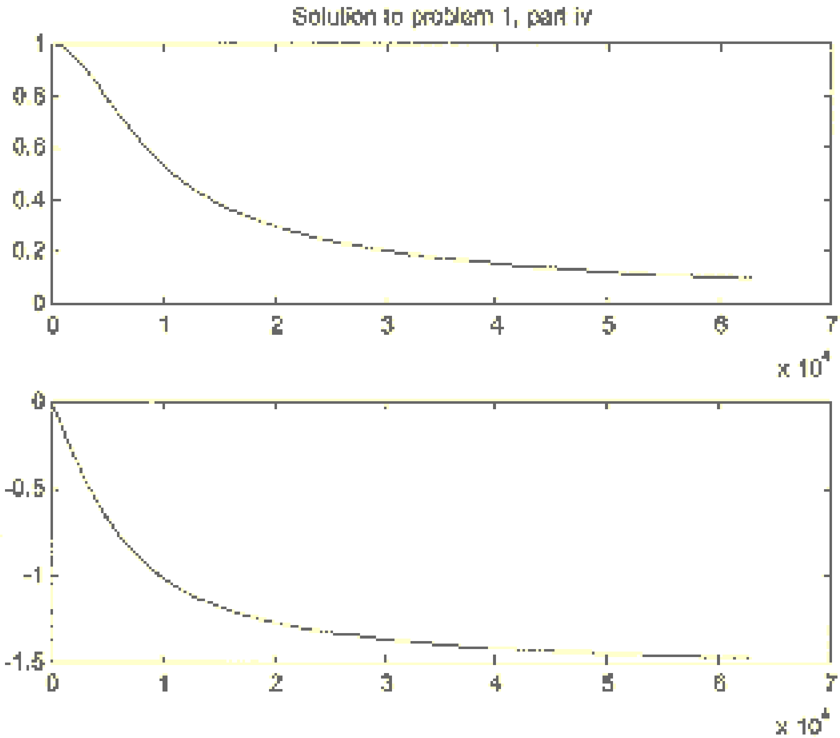


Figure 1

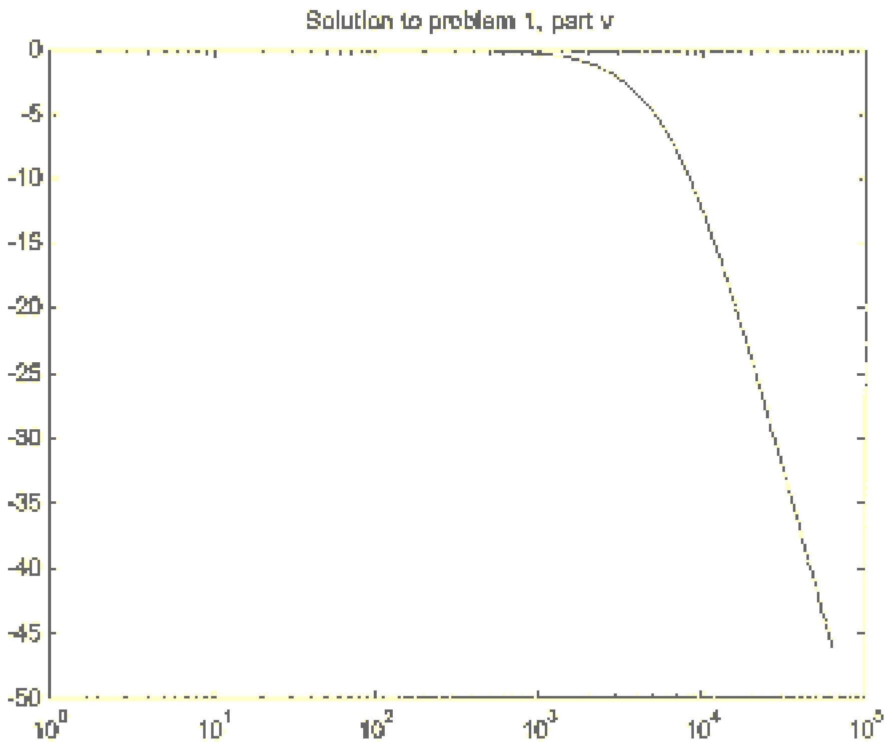


Figure 2

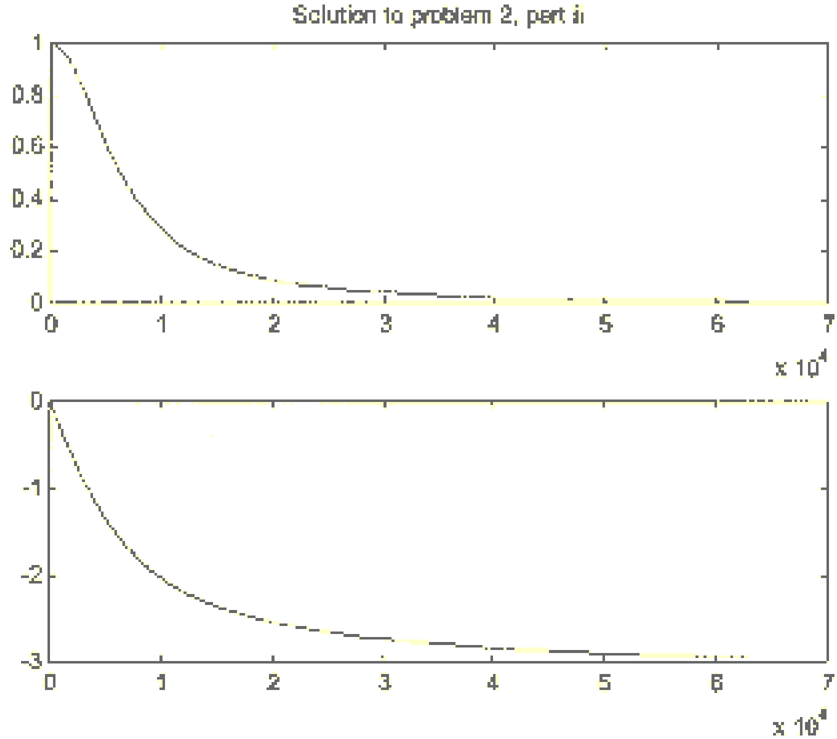


Figure 3.

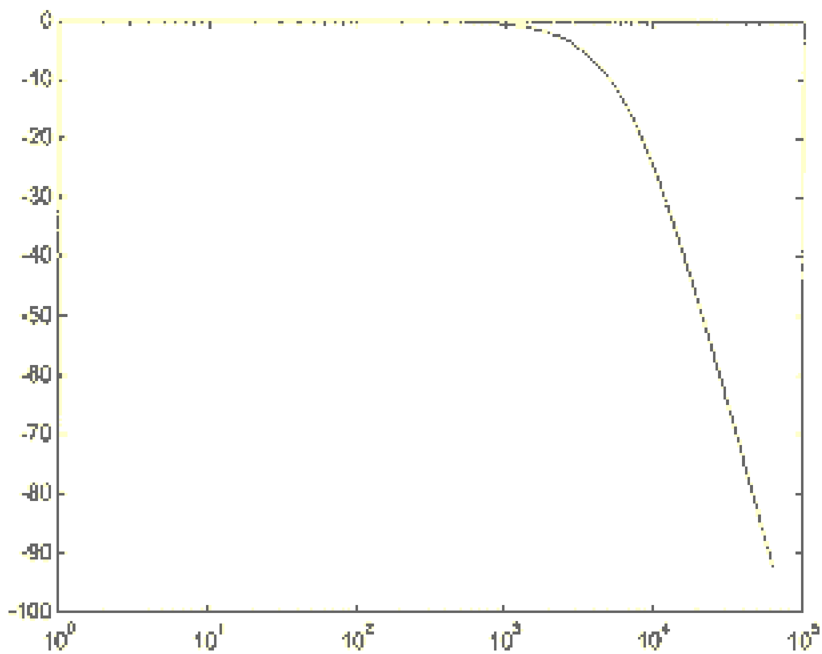


Figure 4.