Now, $Y_k = \frac{1}{p/a}\int_0^{p/a} y(t)e^{-ik(2\pi/p/a)t}\,dt$

$$= \frac{a}{p}\int_0^{p/a} x(at)e^{-ik\left(\frac{2\pi}{p}\right)at}\,dt$$

let $at = u \Rightarrow a\,dt = du$

$\Rightarrow$ If $t=0,\ u=0$

If $t = p/a,\ u = p$

$$\therefore Y_k = \frac{1}{p}\int_{u=0}^{p} x(u)\,e^{-ik\omega_0 u}\,du$$

$$\Rightarrow \boxed{Y_k = X_k}$$

(c) Same as b.

(7) (a) (b) (c) } Use the same techniques as in (6) $\Rightarrow$ NOT HARD!

(8) (a) $x(t) = \begin{cases} 1 & \text{if } t \in [np,\ s+np],\ n \in \text{Int} \\ 0 & \text{elsewhere} \end{cases}$

(b) $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$.

$\omega_0 = \frac{2\pi}{p} \frac{rad}{sec}$

$$X_k = \frac{1}{p}\int_0^p x(t) e^{-ik\omega_0 t}\, dt$$

$$= \frac{1}{p}\left[\int_0^S e^{-ik\omega_0 t}\, dt\right]$$

$$= \frac{1}{p}\left[\frac{e^{-ik\omega_0 t}}{-ik\omega_0}\bigg|_0^S\right]$$

$$= \frac{1}{p}\left[\frac{1}{ik\omega_0}\left(1 - e^{-ik\omega_0 S}\right)\right]$$

$$\Rightarrow X_k = \frac{1}{ikp\omega_0}\left(1 - e^{-ik\omega_0 S}\right)$$

Notice that we cannot determine X_0 from the above formula $\left[0/0\right]$. Therefore, we have to evaluate:

$$X_0 = \frac{1}{P}\int_0^P x(t)\,dt \quad \left[\begin{array}{c} X_0 \text{ is the} \\ \text{DC value} \end{array} \right]$$

$$= \frac{1}{P} \times S \times 1 = \boxed{S/P = X_0}$$

(c) we have already solved this part in (b).

$$Y_k = X_k e^{-ik\omega_0 T}$$

Here, $\omega_0 = \frac{2\pi}{P}$ //

(d) Now, $Z_k = X_k$, but, $\underline{\underline{\omega_z = m\omega_x}}$ (see question (b))

(g) Now, we are given Y_k

i.e. $$Y_k = \frac{1}{P}\int_0^P x(t)\, e^{-ik\omega_0 t}\,dt$$

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{ik\omega_0 t} \quad \text{—— } (2)$$

Now, differentiating (2) wrt t,

$$\dot{y}(t) = \sum_{k=-\infty}^{\infty} Y_k\, ik\omega_0\, e^{ik\omega_0 t}$$

$$\Rightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} Y_k \, i k \omega_0 e^{ik\omega_0 t} \quad \text{—— (4)}$$

Check: $x(t)$ is also periodic with period p.

$$x(t+p) = \frac{d}{dt}\left[y(t+p)\right]$$

$$= \frac{d}{dt}\left[y(t)\right]$$

{ $y(t)$ is periodic with period p $\Rightarrow y(t+p) = y(t)$ }

$$= x(t)$$

$$= \text{R.H.S}$$

$$\therefore \quad x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \quad \text{—— (3)}$$

Comparing (3) and (4)

$$\boxed{X_k = ik \cdot \frac{2\pi}{p} Y_k \quad k \neq 0.}$$

$$X_0 = \frac{1}{p}\int_0^p x(t)\, dt$$

is equal, because, $y(t)$ periodic with period p

$$= \frac{1}{p}\int_0^p \dot{y}(t)\, dt = \frac{1}{p}\left[\cancel{y(p)} - \cancel{y(0)}\right]$$

(Fundamental theorem of calculus)

$$= 0 \quad \text{Ans}$$

2

(7)

(1) mistake! [But straight forward, use the formula]

(1) $x(t+p) = \alpha u(t+p) + \beta v(t+p)$

$= \alpha u(t) + \beta v(t)$ [u & v are periodic with period p]

$= \underline{x(t)}$ ← [You should always check if the function is periodic before applying Fourier series]

$\Rightarrow x(t)$ is periodic with period

Now,

$$X_k = \frac{1}{p}\int_0^p x(t) e^{-ik\omega_0 t}\,dt$$

$$= \frac{1}{p}\int_0^p \left[\alpha u(t) + \beta v(t)\right] e^{-ik\omega_0 t}\,dt$$

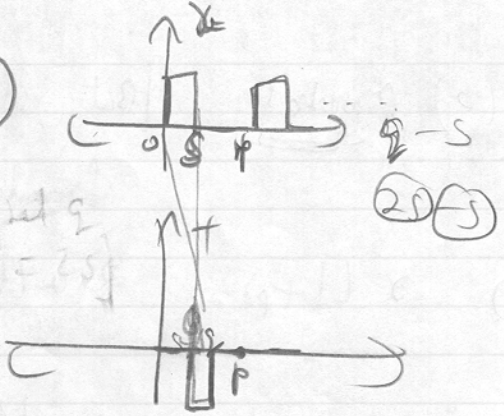
$$= \frac{1}{p}\int_0^p \alpha u(t) e^{-ik\omega_0 t}\,dt + \frac{1}{p}\int_0^p \beta v(t) e^{-ik\omega_0 t}\,dt$$

$$= \alpha\left[\frac{1}{p}\int_0^p u(t) e^{-ik\omega_0 t}\,dt\right] + \beta\left[\frac{1}{p}\int_0^p v(t) e^{-ik\omega_0 t}\right]dt$$

$$\boxed{X_k = \alpha U_k + \beta V_k}$$

[linear combination of F.S coefficients of u and v]

(b) Now, $u(t) = x(t) - x(t-s)$



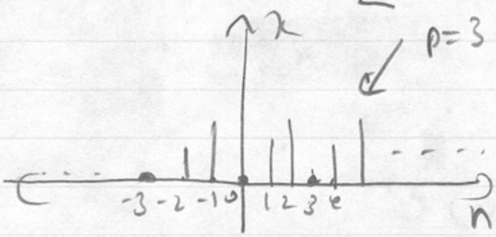
$$U_k = 1 \cdot X_k - X_k e^{-ik\omega_0 s}$$

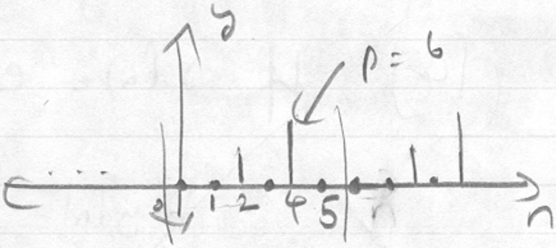
$$\boxed{U_k = X_k(1 - e^{-ik\omega_0 s})}$$

(11)

(12) (a)

Suppose x is





(b) Now, $x(n) = \frac{1}{2}\sum_{k=0}^{2} X_k e^{+ik\omega_0 n}$

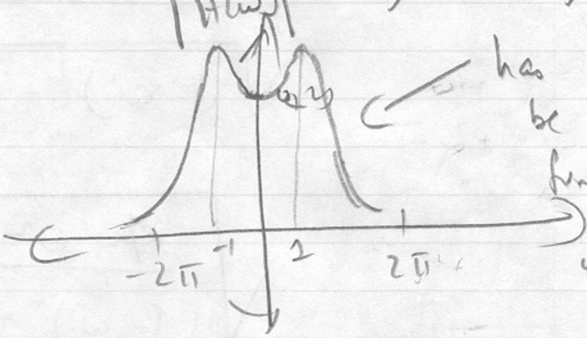
$$y(n) = \begin{cases} x(n/2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases} \Rightarrow$$

(13) Let $x(t) = e^{i\omega t}$

$\Rightarrow y(t) = \hat{H}(\omega) e^{i\omega t}$

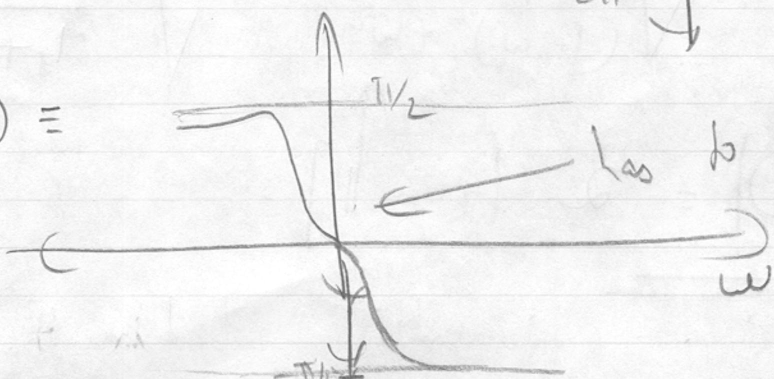
$$-\hat{H}(\omega)\,\omega^2 \cancel{e^{i\omega t}} + \hat{H}(\omega)\, i\omega \cancel{e^{i\omega t}} + \hat{H}(\omega) \cancel{e^{i\omega t}} = i\omega \cancel{e^{i\omega t}} + 2\cancel{e^{i\omega t}}$$

$$\hat{H}(\omega) = \frac{2 + i\omega}{-\omega^2 + i\omega + 1}$$



has to be even function

$\measuredangle \hat{H}(\omega) =$



has to be odd function!

(14) If $x(n) = e^{i\omega n}$,

$y(n) = \hat{H}(\omega) e^{i\omega n}$

$$\therefore \hat{H}(\omega)e^{i\omega(n-2)} + \hat{H}(\omega)e^{i\omega(n-1)} + \hat{H}(\omega)e^{i\omega n}$$

$$= e^{i\omega n} + 2e^{i\omega(n-1)}$$

$$\Rightarrow \boxed{\hat{H}(\omega) = \frac{1+2e^{-i\omega}}{e^{-i\omega 2}+e^{-i\omega}+1}}$$

In order to get this:

$$|\hat{H}(\omega)| = \frac{\sqrt{4+\omega^2}}{\sqrt{(1-\omega^2)^2+\omega^2}}$$

$$= \sqrt{\frac{4+\omega^2}{(1-\omega^2)^2+\omega^2}} = \sqrt{\frac{1+4/\omega^2}{1+\frac{(\omega^2-1)^2}{\omega^2}}} \quad [\text{if } \omega \neq 0!]$$

$$|\hat{H}(0)| = 2 \quad [\text{from } \uparrow]$$

$$\measuredangle H(\omega) = \text{phase angle at the top} - \text{angle at the bottom}$$

$$= \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{1-\omega^2}\right)$$

(14) Just as in (13),

Let $x(n) = e^{i\omega n}$

$\Rightarrow y(n) = H(\omega) e^{i\omega n}$

$$H(\omega)e^{i\omega(n-2)} + H(\omega)e^{i\omega(n-1)} + H(\omega)e^{i\omega n} = e^{i\omega n} + 2e^{i\omega(n-1)}$$

$$\Rightarrow H(\omega)\left[e^{-i\omega 2} + e^{-i\omega} + 1\right] = 1 + 2e^{-i\omega}$$

$$\Rightarrow H(\omega) = \frac{1+2e^{-i\omega}}{1+e^{-i\omega}+e^{-i\omega 2}}$$

$$\therefore |H(\omega)| = \frac{1 + 2\cos(\omega) - 2i\sin(\omega)}{1 + \left[\cos(\omega) - i\sin(\omega)\right] + \left[\cos(2\omega) - i\sin(2\omega)\right]}$$

$$= \frac{\sqrt{[1+2\cos\omega]^2+[2\sin(\omega)]^2}\; e^{i\left[\tan^{-1}\left(\frac{-2\sin\omega}{1+2\cos\omega}\right)\right]}}{\sqrt{[1+\cos(\omega)+\cos(2\omega)]^2+[\sin(\omega)+\sin(2\omega)]^2}\; e^{i\tan^{-1}\left[-\frac{(\sin(\omega)+\sin(2\omega))}{1+\cos(\omega)+\cos 2\omega}\right]}}$$

$$\Rightarrow H(\omega) = \sqrt{\frac{5+4\cos(\omega)}{[1+\cos(\omega)+\cos(2\omega)]^2+[\sin(\omega)+\sin(2\omega)]^2}}\; e^{i\left[\tan^{-1}\left(\frac{-2\sin\omega}{1+2\cos\omega}\right)+\tan^{-1}\left[\frac{\sin(\omega)+\sin 2\omega}{1+\cos(\omega)+\cos(2\omega)}\right]\right]}$$

This is VERY complex. On a midterm, u will be asked SIMPLER questions. We can complete the plot using matlab: ~~However, the function~~

```
>> w= [-pi/2 : 0.01 : pi/2]   % Graph has asymptotes!
>> H = (1+ 2*exp(-i*w)) ./ (1+ exp(-i*w) +exp(-i*w*2));
>> subplot(2,1,1), plot(w, abs(H))
>> subplot(2,1,2), plot(w, angle(H))
```