EECS 20. Midterm 2 Solution
November 9, 2001.

1. **20 points.** Consider a continuous-time signal x: *Reals* $\to$ *Reals* defined by

$$\forall\, t \in \textit{Reals}, \quad x(t) = \cos(\omega_1 t) + \cos(\omega_2 t),$$

where $\omega_1 = 2\pi$ and $\omega_2 = 3\pi$ radians/second.

   (a) Find the smallest period $p \in \textit{Reals}_+$, where $p > 0$.
   Solution: $p = 2$.

   (b) Give the fundamental frequency corresponding to the period in (a). Give the units.
   Solution: $\omega_0 = \pi$ radians/second.

   (c) Give the coefficients $A_0, A_1, A_2, \cdots$ and $\phi_1, \phi_2, \cdots$ of the Fourier series expansion for x.
   Solution: $A_2 = A_3 = 1$, $A_k = 0$, $\forall\, k \notin \{2, 3\}$, and $\phi_k = 0$, $\forall\, k \in$ *Naturals*.

2. **30 points.** Suppose that the continuous-time signal x: *Reals* $\to$ *Reals* is periodic with period p. Let the fundamental frequency be $\omega_0 = 2\pi/p$. Suppose that the Fourier series coefficients for this signal are known constants $A_0, A_1, A_2, \cdots$ and $\phi_1, \phi_2, \cdots$. Give the Fourier series coefficients $A'_0, A'_1, A'_2, \cdots$ and $\phi'_1, \phi'_2, \cdots$ for each of the following signals:

   (a) ax, where $a \in$ *Reals* is a constant;
   Solution: $A'_k = aA_k$, $\forall\, k \in \textit{Naturals}_0$ and $\phi'_k = \phi_k$, $\forall\, k \in$ *Naturals*.

   (b) $D_\tau(x)$, where $\tau \in$ *Reals* is a constant; and
   Solution: $A'_k = A_k$, $\forall\, k \in \textit{Naturals}_0$ and $\phi'_k = \phi_k - k\omega_0\tau$, $\forall\, k \in$ *Naturals*.

   (c) $S(x)$, where S is an LTI system with frequency response H given by

$$\forall\, \omega \in \textit{Reals}, \quad H(\omega) = \begin{cases} 1; & \text{if } \omega = 0 \\ 0; & \text{otherwise} \end{cases}$$

   (Note that this is a highly unrealistic frequency response.)
   Solution: $A'_0 = A_0$, $A'_k = 0$, $\forall\, k \in$ *Naturals* and $\phi'_k = 0$, $\forall\, k \in$ *Naturals*. (Any other value for ϕ'_k is acceptable.
   Extra Credit:

   (d) Let y: *Reals* $\to$ *Reals* be another periodic signal with period p. Suppose y has Fourier series coefficients $A''_0, A''_1, A''_2, \cdots$ and $\phi''_1, \phi''_2, \cdots$. Give the Fourier series coefficients of $x + y$.
   Solution:

$$\forall\, k \in \textit{Naturals}_0, \quad A'_k = |A_k e^{i\phi_k} + A''_k e^{i\phi''_k}|,$$

   and

$$\forall\, k \in \textit{Naturals}, \quad \phi'_k = \angle(A_k e^{i\phi_k} + A''_k e^{i\phi''_k}).$$

3. **30 points.** Consider discrete-time systems with input x: *Integers* $\to$ *Reals* and output y: *Integers* $\to$ *Reals*. Each of the following defines such a system. For each, indicate whether it is linear (L), time-invariant (TI), both (LTI), or neither (N). Note that no partial credit will be given for these questions.

   (a) $\forall\, n \in$ *Integers*, $\quad y(n) = x(n) + 0.9y(n-1)$
   Solution: LTI

   (b) $\forall\, n \in$ *Integers*, $\quad y(n) = \cos(2\pi n)x(n)$
   Solution: LTI

   (c) $\forall\, n \in$ *Integers*, $\quad y(n) = \cos(2\pi n/9)x(n)$
   Solution: L

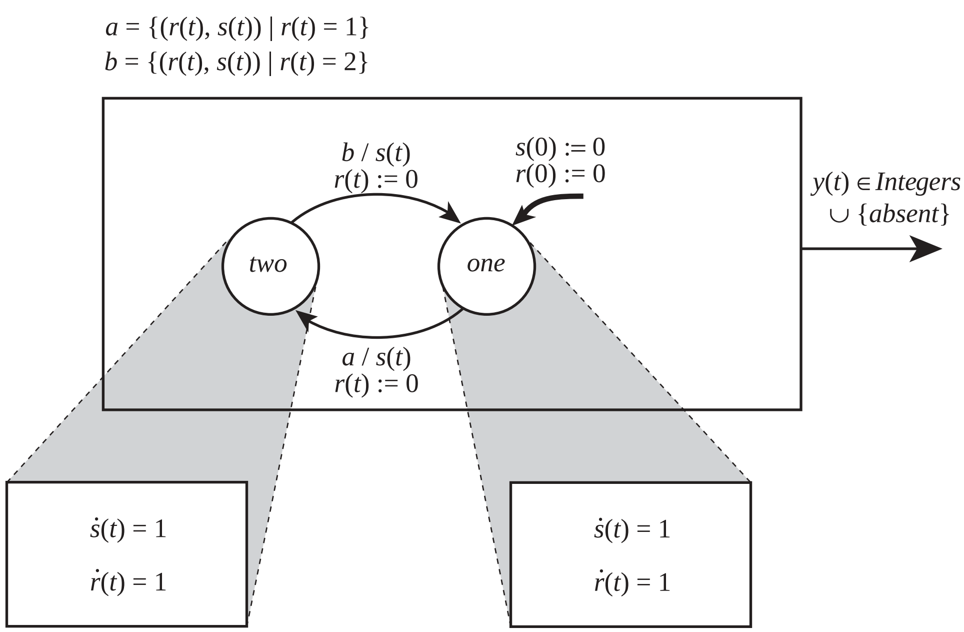
   (d) $\forall\, n \in$ *Integers*, $\quad y(n) = \cos(2\pi n/9)(x(n) + x(n-1))$
   Solution: L

   (e) $\forall\, n \in$ *Integers*, $\quad y(n) = x(n) + 0.1(x(n))^2$
   Solution: TI

   (f) $\forall\, n \in$ *Integers*, $\quad y(n) = x(n) + 0.1(x(n-1))^2$
   Solution: TI

4. **20 points.** The objective of this problem is to understand a timed automaton, and then to modify it as specified.

   (a) For the timed automaton shown below, describe the output y. You will lose points for imprecise or sloppy notation.



   Solution: The system generates an event sequence

$$(1, 3, 4, 6, 7, 9, 10, \cdots)$$

at times

$$1, 3, 4, 6, 7, 9, 10, \cdots .$$

That is, the value of each output event is equal to the time at which it is produced, and the intervals between events alternate between one and two seconds. Precisely,

$$y(t) = \begin{cases} t & \text{if } t = 3k \text{ for some } k \in \textit{Naturals} \\ t & \text{if } t = 3k+1 \text{ for some } k \in \textit{Naturals} \\ \textit{absent} & \text{otherwise} \end{cases}$$

(b) Assume there is a new input $u \colon \textit{Reals} \to \textit{Inputs}$ with alphabet

$$\textit{Inputs} = \{\textit{reset}, \textit{absent}\},$$

and that when the input has value *reset*, the hybrid system starts over, behaving as if it were starting at time 0 again. Modify the diagram below so that it behaves like the system in (a) except that it responds to the *reset* input accordingly. Again, you will lose point for imprecise or sloppy notation.