EECS 20. Final Exam Practice Problems, December 12, 2004.

There will be additional problems, as time permits.

1. Give a precise definition of the space *Images* of all 600×900 grayscale images with pixel values represented as 8-bit integers. Let $Bin^N = \{0,1\}^N$ be the space of all 0-1 sequences of length N. Define a system

   $$Coder : Images \to Bin^N,$$

   such that (1) the function *Coder* is one-to-one and (2) N is as small as possible.

2. Find a function

   $$f : [Ints \to Reals^2] \to [Ints \to Complex]$$

   that is linear, one-to-one and onto. Prove that your choice of f has these properties.

3. What will the following Matlab code produce?

```
>> k = 0:199;
>> x = (sin(k*2*pi/50 + pi/2) +1)';
>> c = 128 * repmat(x, 200, 1);
>> image(c), axis image
```

4. Design a state machine with $Inputs = \{0,1\}$, $Outputs = \{T,F\}$ such that if S denotes the state machine's input-output function,

   $$\forall x, \forall n, \quad S(x)(n) = \begin{cases} T, & \text{if (number of 0's)} - \text{(number of 1's) in } x_0, \cdots, x_n = 2 \\ F, & \text{otherwise} \end{cases}$$

   Prove that there is no *finite* state machine that realizes S.

5. Design a virtual cat as a state machine (i.e. specify its inputs, outputs, etc.) that behaves as follows:

   > It starts out *happy*. If you *pet* it, it *purrs*. If you *feed* it, it *throws up*. If *time passes*, it gets *hungry* and *rubs* against your legs. If you feed it when it is hungry, it sometimes purrs and gets happy, and sometimes it stays hungry and rubs against your legs. If you pet it when it is hungry, it *bites* you. If time passes when it is hungry, it *dies*.

   Is your machine deterministic or non-deterministic?

6. Determine if each of the following statements is true or false. Provide a proof or counter-example to support your answer.

(a) If a deterministic state machine M is placed in feedback composition, the result is always well-formed.

(b) If a non-deterministic machine N simulates a deterministic machine M, then M simulates N.

(c) If machine M has n states, then every state that is reachable from the initial state can be reached by an input sequence of length at most $n-1$.

(d) If machines M_k has n_k states, $k = 1, 2$, the cascade composition of M_1 and M_2 has $n_1 + n_2$ states.

7. Consider the difference equation

$$y(n) - y(n-1) = x(n) - 2x(n-1).$$

(a) Take the state at time n as $s(n) = [y(n-1), x(n-1)]^T$ and write down the $[A, b, c, d]$ representation of the system. Find its zero-state impulse response.

(b) Implement the difference equation using two delay elements whose outputs are the two state components.

(c) Find another implementation using only *one* delay element. Write the $[A, b, c, d]$ representation for this implementation. Find its zero-state impulse response.

(d) Are the two impulse responses the same?

(e) Find the frequency response directly from the difference equation and by taking the DTFT of the impulse response. Are the two frequency responses the same?

(f) Sketch the magnitude and phase response.

8. Consider a LTI system with $[A, b, c, d]$ representation given by:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^T = [1 \;\; 1], \quad d = 0.$$

(a) Suppose the initial state is $s(0) = [0 \;\; 0]^T$. Find an input sequence $x(0), x(1)$ of length two such that the state at time 2 is $s(2) = [1 \;\; 1]^T$.

(b) Suppose the initial state is $s(0) = [s_1 \;\; s_2]^T$. Find an input sequence $x(0), x(1)$ such that the state at time 2 is $s(2) = [0 \;\; 0]^T$. (The input sequence will depend on $s(0)$.)

9. Two SISO systems with representations $[A_i, b_i, c_i, d_i], i = 1, 2$ are connected in cascade composition. Find an $[A, b, c, d]$ representation for the composiiton.

10. A discrete-time, causal LTI system S produces the output y given by

$$y(n) = \delta(n) + \delta(n-1) + \delta(n-2),$$

in response to the input x given by

$$x(n) = \delta(n) + \delta(n-2).$$

Find the impulse reponse h of S.

11. A linear system with input x and output y is described by the second order differential equation,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t). \tag{1}$$

(a) Find the frequency response of this system. Give simple expressions for: $\forall \omega$,
$H(\omega) =$
$|H(\omega)| =$
$\angle H(\omega) =$

(b) Find the partial fraction expansion of H, i.e. determine the constants a, b, A, B in the formula

$$H(\omega) = \frac{1}{(i\omega + a)(i\omega + b)} = \frac{A}{i\omega + a} + \frac{B}{i\omega + b}. \tag{2}$$

Next find the (zero-state) impulse response of the system (1) by taking the inverse Fourier Transform of H using (2).

(c) Now find the (zero-state) step response of the system (1).

12. If $x(n) = n, n = 0, 1, 2, 3$, find its 4-point DFT.

13. Find the DTFT of the signal

$$\forall n \in \textit{Ints}, x(n) = (0.5)^{|n|}$$

14. Recall the inverse DTFT formula

$$\forall n, \quad x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{i\omega n} d\omega.$$

(a) Use this formula to guess and verify the DTFT of the signal $n \mapsto e^{j\omega_0 n}$, where $0 \le \omega_0 < 2\pi$.

(b) What is the DTFT of the signal $n \mapsto \cos(\omega_0 n)$ for $0 \le \omega_0 < 2\pi$.

(c) What is the DTFT of the signal $n \mapsto e^{i\omega_0 n}$ for $\omega_0 = 2\pi + \pi/4$. Note: $\omega_0 > 2\pi$.

15. Recall that the Fourier Transform (FT) of $x \in$ *ContSignals* is $X \in$ *ContSignals* given by

$$\forall \omega, \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt. \tag{3}$$

(a) Differentiate both sides of (3) with respect to ω n times and prove that the FT of the signal $t \mapsto t^n x(t)$ is

$$\omega \mapsto (i)^n X^{(n)}(\omega) = (i)^n \frac{d^n X}{d\omega^n}(\omega).$$

(b) Find the FT of the signal $e^{-t}u(t)$.

(c) Find the FT of the signal $t^n e^{-t} u(t)$.

16. The bandwidth of a continuous time signal x with FT X is by definition the smallest frequency ω_B such that $X(\omega) = 0$ for $|\omega| > \omega_B$.

    (a) What is the bandwidth of the signals: $\forall t \in$ *Reals*,

    $$x_k(t) = \cos(10k\pi t), k = 1, 2, 3; \quad x_4(t) = x_1(t) + x_2(t) + x_3(t).$$

    (b) What is the FT of $x_k, k = 1, \cdots, 4$?

    (c) Suppose x_k is sampled at frequency $\omega_s = 30\pi$ rad/sec. Find a simple expression for the sampled signal y_k.

    (d) Find signals z_k : *Reals* $\rightarrow$ *Reals* such that (i) the bandwidth of z_k is smaller than 15π rad/sec, which is one-half the sampling frequency; and (ii) if z_k is sampled at frequency ω_s it also yields the signal y_k.

17. A continuous signal $x_a(t)$ has the Fourier Transform

    $$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

    such that $X_a(\Omega) = 0, |\Omega| > 2\pi \times 1$ rad/sec.

    (a) Express the discrete time Fourier transform

    $$X_b(\omega) = \sum_{n=-\infty}^{\infty} x_b(n) e^{-j\omega n}$$

    in terms of $X_a(\Omega)$,where $x_b(n) = x_a(n + 0.25), \forall n \in Intergers.$

    (b) Can the continuous time Fourier transform $X_a(\Omega)$ be uniquely determined from $X_b(W)$? If yes, how; if not, why not?

    (c) Now assume that in addition to $x_b(n)$, another set of samples $x_c(n) = x_a(n)$ is obtained. Can $x_a(t)$ be uniquely determined from $x_b(n)$ and $x_c(n)$? If yes, how; if not, why not?

    (d) What conclusions might you draw about sampling of bandlimited signals on the basis of your results?

18. The autocorrelation sequence of a signal $X(n)$ is defined as

    $$R_x(n) = \sum_{k=-\infty}^{\infty} x^*(k) x(n + k)$$

    Express the Fourier transform of $R_x(n)$ in terms of $X(\omega)$, the Fourier transform of $x(n)$.