EECS 20. Final Exam Practice Problems, December 12, 2004.

There will be additional problems, as time permits.

1. Give a precise definition of the space *Images* of all 600×900 grayscale images with pixel values represented as 8-bit integers. Let $Bin^N = \{0,1\}^N$ be the space of all 0-1 sequences of length N. Define a system

$$Coder : Images \to Bin^N,$$

such that (1) the function *Coder* is one-to-one and (2) N is as small as possible.

Answer to 1

$$Images = [V \times H \to Bin^8],$$

with $V = \{1, \cdots, 600\}$, $H = \{1, \cdots, 900\}$. Take $N = 600 \times 900 \times 8$. If the (i, j)th pixel value of $image \in Images$ is $image(i, j) = (b_1, \cdots, b_8) \in Bin^8$, denote $b_k = image(i, j)(k)$.
Let

$$f : \{1, \cdots, N\} \to V \times H \times \{1, \cdots, 8\}$$

be any one-to-one (and onto) function. Define $Coder : Images \to Bin^N$ by

$$\forall 1 \le n \le N, \quad Coder(image)(n) = image(i, j)(k),$$

in which $f(n) = (i, j, k)$.

2. Find a function

$$f : [Ints \to Reals^2] \to [Ints \to Complex]$$

that is linear, one-to-one and onto. Prove that your choice of f has these properties.

Answer to 2 For $x = (x_1, x_2) \in [Ints \to Reals^2]$, define

$$f(x)(n) = x_1(n) + ix_2(n) \in Complex.$$

Check that $f(ax + by) = af(x) + bf(y)$ for $a, b \in Reals$, so f is linear. Next, note that if

$$f(x)(n) = x_1(n) + ix_2(n) = f(y)(n) = y_1(n) + iy_2(n),$$

then $x(n) = y(n)$, so that f is one-to-one. Finally, for any $z \in [Ints \to Complex]$, $z = f(x)$ for x given by

$$x(n) = (Rez(n), Imz(n)),$$

so that f is onto.

3. What will the following Matlab code produce?

```
>> k = 0:199;
>> x = (sin(k*2*pi/50 + pi/2) +1);
>> c = 128 * repmat(x, 200, 1);
>> image(c), axis image
```

Answer to 3 The code will produce a 200×200 pixel image whose intensity varies sinusoidally in the vertical direction.

4. Design a state machine with $Inputs = \{0, 1\}$, $Outputs = \{T, F\}$ such that if S denotes the state machine's input-output function,

$$\forall x, \forall n, \quad S(x)(n) = \begin{cases} T, & \text{if (number of 0's)} - \text{(number of 1's) in } x_0, \cdots, x_n = 2 \\ F, & \text{otherwise} \end{cases}$$

Prove that there is no *finite* state machine that realizes S.

Answer to 4 Proof by contradiction. Assume that there is a machine with $n(< \infty)$ states that realizes the input-output function. Consider the input sequence 0^n and the machine's state response

$$s(0), s(1), \cdots, s(n).$$

Since there are n states, there exist $0 \leq i < j \leq n$ with $s(i) = s(j)$.

But then the outputs at the end of the two input sequences

$$0^{i+2}1^i \text{ and } 0^{j+2}1^i$$

must be the same, which contradicts the assumption.

5. Design a virtual cat as a state machine (i.e. specify its inputs, outputs, etc.) that behaves as follows:

> It starts out *happy*. If you *pet* it, it *purrs*. If you *feed* it, it *throws up*. If *time passes*, it gets *hungry* and *rubs* against your legs. If you feed it when it is hungry, it sometimes purrs and gets happy, and sometimes it stays hungry and rubs against your legs. If you pet it when it is hungry, it *bites* you. If time passes when it is hungry, it *dies*.

Is your machine deterministic or non-deterministic?

Answer to 5 The transition diagram of the state machine is given in figure 1, from which one can read of the inputs, outputs, etc. The machine is non-deterministic. [Note: the else self-loop is not shown.]

6. Determine if each of the following statements is true or false. Provide a proof or counterexample to support your answer.

   (a) If a deterministic state machine M is placed in feedback composition, the result is always well-formed.

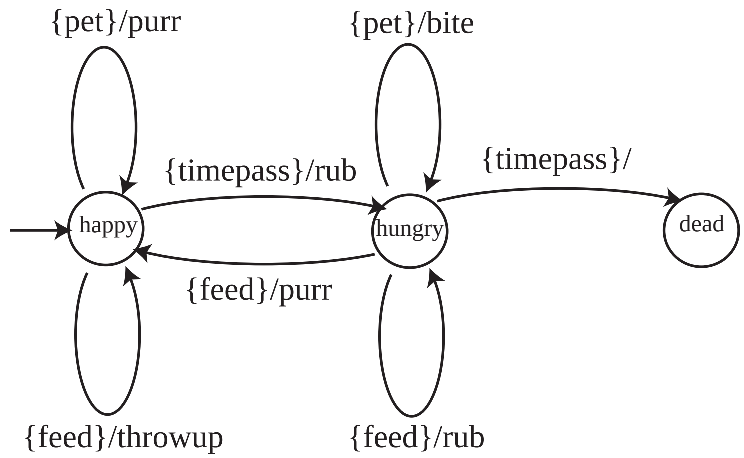

Figure 1: State machine for problem 5

(b) If a non-deterministic machine N simulates a deterministic machine M, then M simulates N.

(c) If machine M has n states, then every state that is reachable from the initial state can be reached by an input sequence of length at most $n-1$.

(d) If machines M_k has n_k states, $k = 1, 2$, the cascade composition of M_1 and M_2 has $n_1 + n_2$ states.

Answer to 6 (a) No. There is a 2-state machine counter-example in the text.

(b) No. Take any non-deterministic state machine N whose behaviors do not correspond to a function. N cannot be simulated by any deterministic machine since the latter's behaviors form a graph of its input-output function.

(c) Yes.

(d) No. The cascade composition has $n_1 \times n_2$ states.

7. Consider the difference equation

$$y(n) - y(n-1) = x(n) - 2x(n-1).$$

(a) Take the state at time n as $s(n) = [y(n-1), x(n-1)]^T$ and write down the $[A, b, c, d]$ representation of the system. Find its zero-state impulse response.

(b) Implement the difference equation using two delay elements whose outputs are the two state components.

(c) Find another implementation using only *one* delay element. Write the $[A, b, c, d]$ representation for this implementation. Find its zero-state impulse response.

(d) Are the two impulse responses the same?

(e) Find the frequency response directly from the difference equation and by taking the DTFT of the impulse response. Are the two frequency responses the same?

(f) Sketch the magnitude and phase response.

Answer to 7 (a) From the update equations

$$\begin{aligned} s(n+1) = \begin{bmatrix} y(n) \\ x(n) \end{bmatrix} &= \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(n), \\ y(n) &= [1 \quad -2] \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + [1]\, x(n), \end{aligned}$$

one can read off A, b, c, d. The frequency response can be read off from the difference equation,

$$H(\omega) = \frac{1 - 2e^{-i\omega}}{1 - e^{-i\omega}} = \frac{1}{1 - e^{-i\omega}} - 2\frac{e^{-i\omega}}{1 - e^{-i\omega}}.$$

The impulse response h is the inverse DTFT of H which we can write down as

$$\forall n, \quad h(n) = u(n) - 2u(n-1),$$

in which u is the step: $u(n) = 0, n < 0, u(n) = 1, n \geq 0$. [Alternatively, one can find h from A, b, c, d, as seen next.]

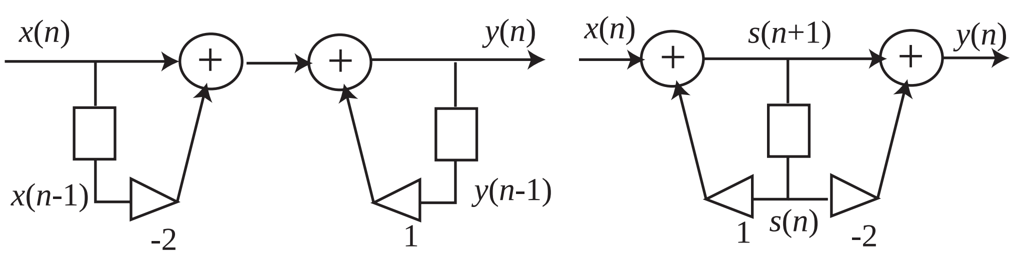

Figure 2: Implementations for problem 7

(b, c, d) The two implementations are shown in figure 2. The second implementation yields

$$\begin{aligned} s(n+1) &= s(n) + x(n) \\ y(n) &= s(n+1) - 2s(n) \\ &= -s(n) + x(n), \end{aligned}$$

so its zero-state impulse response is

$$h(n) = \begin{cases} d = 1, & n = 0 \\ c^T A^{n-1} b = -1, & n \geq 1 \end{cases} = u(n) - 2u(n-1)$$

as obtained before.

(e,f) We have seen H above, from which

$$\begin{aligned} H(\omega) &= \frac{1 - 2\cos(\omega) + i2\sin(\omega)}{1 - \cos(\omega) + i\sin(\omega)} \\ |H(\omega)| &= \frac{|1 - 2\cos(\omega) + i2\sin(\omega)|}{|1 - \cos(\omega) + i\sin(\omega)|} \\ \angle H(\omega) &= \tan^{-1}\frac{2\sin(\omega)}{1 - 2\cos(\omega)} - \tan^{-1}\frac{\sin(\omega)}{1 - \cos(\omega)} \end{aligned}$$

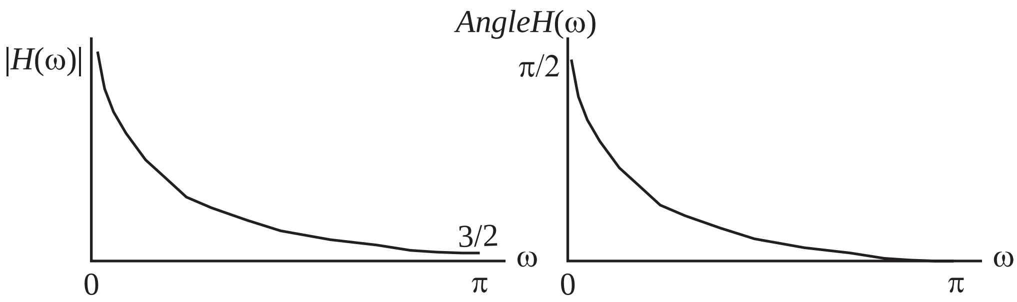

Figure 3: Plots problem 7

Figure 3 gives the plots. Observe that for $\omega > 0$ small,

$$H(\omega) \approx \frac{-1+i2\omega}{i\omega} \approx \frac{i}{\omega},$$

so as

$$\omega \to 0+, \quad |H(\omega)| \to \infty \ \angle H(\omega) \to \frac{\pi}{2}.$$

For

$$\omega = \pi, \quad H(\omega) = \frac{3}{2}.$$

8. Consider a LTI system with $[A, b, c, d]$ representation given by:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^T = [1 \ \ 1], \quad d = 0.$$

   (a) Suppose the initial state is $s(0) = [0 \ \ 0]^T$. Find an input sequence $x(0), x(1)$ of length two such that the state at time 2 is $s(2) = [1 \ \ 1]^T$.
   (b) Suppose the initial state is $s(0) = [s_1 \ \ s_2]^T$. Find an input sequence $x(0), x(1)$ such that the state at time 2 is $s(2) = [0 \ \ 0]^T$. (The input sequence will depend on $s(0)$.)

Answer to 8 (a) We have

$$\begin{aligned} s(2) &= A^2 s(0) + Abx(0) + bx(1) \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} s(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(1) \\ &= \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \text{ for } s(0) = 0 \end{aligned}$$

So $x(0) = s_1(2) = 1, x(0) = s_2(2) = 1$.

(b) From above,

$$s(2) = \begin{bmatrix} 2s_1 + s_2 + x(0) \\ s_1 + s_2 + x(1) \end{bmatrix},$$

so $x(0) = -2s_1 - s_2$ and $x(1) = -s_1 - s_2$.

9. Two SISO systems with representations $[A_i, b_i, c_i, d_i], i = 1,2$ are connected in cascade composition. Find an $[A,b,c,d]$ representation for the composiiton.

   Answer to 9 Let the two systems be

$$\begin{aligned} s_i(n+1) &= A_i s_i(n) + b_i x_i(n) \\ y_i(n) &= c_i^T s_i(n) + d_i x_i(n). \end{aligned}$$

   Because of the cascade connection $x_2(n) = y_1(n)$, so

$$\begin{aligned} s_2(n+1) &= A_2 s_2(n) + b_2 x_2(n) = A_2 s_2(n) + b_2[c_1^T s_1(n) + d_1 x_1(n)] \\ y_2(n) &= c_2^T s_2(n) + d_2 x_2(n) = c_2^T s_2(n) + d_2[c_1^T s_1(n) + d_1 x_1(n)]. \end{aligned}$$

   So the representation for the composite system is

$$\begin{aligned} \begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ b_2 c_1^T & A_2 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 d_1 \end{bmatrix} x_1(n) \\ y_2(n) &= \begin{bmatrix} d_2 c_1^T & c_2^T \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + [d_2 d_1] x_1(n) \end{aligned}$$

10. A discrete-time, causal LTI system S produces the output y given by

$$y(n) = \delta(n) + \delta(n-1) + \delta(n-2),$$

   in response to the input x given by

$$x(n) = \delta(n) + \delta(n-2).$$

   Find the impulse reponse h of S.

   Answer to 10 Let $h : Ints \to Reals$ be the impulse response. Because S is causal, $h(n) = 0, n < 0$. The response to the input $x(n) = \delta(n) + \delta(n-2)$ is

$$\begin{aligned} y(n) &= h(n) + h(n-2) = \delta(n) + \delta(n-1) + \delta(n-2) \\ &= \begin{cases} 1, & n = 0,1,2 \\ 0, & n > 2 \end{cases} \end{aligned}$$

   Evaulating, gives

$$\begin{aligned} h(0) + h(-2) &= h(0) = 1 \\ h(1) + h(-1) &= h(1) = 1 \\ h(2) + h(0) &= h(2) + 1 = 1 \Longrightarrow h(2) = 0 \\ h(3) + h(1) &= h(3) + 1 = 0 \Longrightarrow h(3) = -1 \\ h(4) + h(2) &= h(4) = 0 \\ h(5) + h(3) &= h(5) - 1 = 0 \Longrightarrow h(5) = 1 \end{aligned}$$

Continuing in this way one gets

$$(h(0), h(1), h(2), \cdots) = (1, 1, 0, -1, 0, 1, 0, -1, 0, 1, \cdots).$$

11. A linear system with input x and output y is described by the second order differential equation,

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t). \tag{1}$$

   (a) Find the frequency response of this system. Give simple expressions for: $\forall \omega$,
   $H(\omega) =$
   $|H(\omega)| =$
   $\angle H(\omega) =$

   (b) Find the partial fraction expansion of H, i.e. determine the constants a, b, A, B in the formula

$$H(\omega) = \frac{1}{(i\omega + a)(i\omega + b)} = \frac{A}{i\omega + a} + \frac{B}{i\omega + b}. \tag{2}$$

   Next find the (zero-state) impulse response of the system (1) by taking the inverse Fourier Transform of H using (2).

   (c) Now find the (zero-state) step response of the system (1).

Answer to 11 (a) We have

$$H(\omega) = \frac{1}{-\omega^2 + 2 + i3\omega};\ |H(\omega)| = \frac{1}{[(2-\omega^2)^2 + 9\omega^2]^{1/2}};\ \angle H(\omega) = -\tan^{-1}\frac{3\omega}{2-\omega^2}.$$

(b) The partial fraction expansion is

$$H(\omega) = \frac{1}{(i\omega + 1)(i\omega + 2)} = \frac{1}{i\omega + 1} - \frac{1}{i\omega + 2},$$

whose inverset FT, using the tables, is

$$\forall t \in Reals, \quad h(t) = [e^{-t} - e^{-2t}]u(t).$$

(c) The step response is the response to the step input u,

$$\begin{aligned} \forall t, \quad s(t) &= \int_{-\infty}^{\infty} h(s)u(t-s)ds = \int_0^t h(s)ds \\ &= \int_0^t [e^{-s} - e^{-2s}]ds = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}. \end{aligned}$$

12. If $x(n) = n, n = 0, 1, 2, 3$, find its 4-point DFT.

13. Find the DTFT of the signal

$$\forall n \in Ints, x(n) = (0.5)^{|n|}$$

Answer to 13 The DTFT is:

$$\begin{aligned}
\forall \omega, \quad X(\omega) &= \sum_{-\infty}^{\infty} x(n)e^{-i\omega n} = \sum_{0}^{\infty}(0.5)^n e^{-i\omega n} + \sum_{-\infty}^{1}(0.5)^{-n}e^{-i\omega n} \\
&= \sum_{0}^{\infty}(0.5e^{-i\omega})^n + \sum_{0}^{\infty}(0.5e^{i\omega})^n - 1 \\
&= \frac{1}{1-0.5e^{-i\omega}} + \frac{1}{1-0.5e^{i\omega}} - 1 \\
&= \frac{2-\cos(\omega)}{1.25-\cos(\omega)} - 1 = \frac{0.75}{1.25-\cos(\omega)}
\end{aligned}$$

14. Recall the inverse DTFT formula

$$\forall n, \quad x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{i\omega n}d\omega.$$

(a) Use this formula to guess and verify the DTFT of the signal $n \mapsto e^{i\omega_0 n}$, where $0 \le \omega_0 < 2\pi$.

(b) What is the DTFT of the signal $n \mapsto \cos(\omega_0 n)$ for $0 \le \omega_0 < 2\pi$.

(c) What is the DTFT of the signal $n \mapsto e^{i\omega_0 n}$ for $\omega_0 = 2\pi + \pi/4$. Note: $\omega_0 > 2\pi$.

Answer to 14 (a) Since $\int_0^{2\pi}\delta(\omega-\omega_0)e^{i\omega n}d\omega = e^{i\omega_0 n}$, we conclude that $X(\omega) = 2\pi\delta(\omega-\omega_0)$.

(b) By linearity

$$\cos(\omega_0 n) \leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

(c) Since $n \mapsto e^{i(2\pi+\pi/4)n} = e^{i\pi/4n}$, its DTFT is $2\pi\delta(\omega-\pi/4)$.

15. Recall that the Fourier Transform (FT) of $x \in ContSignals$ is $X \in ContSignals$ given by

$$\forall \omega, \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt. \tag{3}$$

(a) Differentiate both sides of (3) with respect to ω n times and prove that the FT of the signal $t \mapsto t^n x(t)$ is

$$\omega \mapsto (i)^n X^{(n)}(\omega) = (i)^n \frac{d^n X}{d\omega^n}(\omega).$$

(b) Find the FT of the signal $e^{-t}u(t)$.

(c) Find the FT of the signal $t^n e^{-t}u(t)$.

Answer to 15 (a) Differentiation with respect to ω n times gives

$$X^{(n)}(\omega) = \int_{-\infty}^{\infty} (-it)^n e^{-i\omega t} x(t) dt,$$

from which

$$i^n X^{(n)}(\omega) = \int_{\infty}^{\infty} t^n x(t) e^{-i\omega t} dt,$$

so that indeed

$$t^n x(t) \leftrightarrow i^n X^{(n)}(\omega).$$

(b) The FT is

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} e^{-t} u(t) e^{-i\omega t} dt = \int_0^{\infty} e^{-t} e^{-i\omega t} dt \\ &= \frac{1}{1+i\omega}. \end{aligned}$$

(c) From calculus one can check that

$$\frac{d^n}{d\omega^n}[1+i\omega]^{-1} = (-i)^n n! [1+i\omega]^{-(n+1)}.$$

Hence

$$t^n e^{-t} u(t) \leftrightarrow \frac{n!}{(1+i\omega)^n}.$$

16. The bandwidth of a continuous time signal x with FT X is by definition the smallest frequency ω_B such that $X(\omega) = 0$ for $|\omega| > \omega_B$.

   (a) What is the bandwidth of the signals: $\forall t \in Reals$,

   $$x_k(t) = \cos(10k\pi t), k = 1, 2, 3; \quad x_4(t) = x_1(t) + x_2(t) + x_3(t).$$

   (b) What is the FT of $x_k, k = 1, \cdots, 4$?

   (c) Suppose x_k is sampled at frequency $\omega_s = 30\pi$ rad/sec. Find a simple expression for the sampled signal y_k.

   (d) Find signals $z_k : Reals \to Reals$ such that (i) the bandwidth of z_k is smaller than 15π rad/sec, which is one-half the sampling frequency; and (ii) if z_k is sampled at frequency ω_s it also yields the signal y_k.

Answer to 16 (a) The bandwidth of x_k is $10k\pi$ rad/sec, $k = 1, 2, 3$; the bandwidth of x_4 is 30π rad/sec.

(b) The FT are:

$$\begin{aligned} x_k &\leftrightarrow X_k(\omega) = \pi[\delta(\omega - 10k\pi) + \delta(\omega + 10k\pi)]; k = 1, 2, 3 \\ x_4 &\leftrightarrow X_4 = X_1 + X_2 + X_3. \end{aligned}$$

(c) We have

$$
\begin{aligned}
y_k(n) &= x_k(n \times \frac{2}{30}) \\
&= \begin{cases} \cos(\frac{2}{3}\pi n), & k=1 \\ \cos(\frac{4}{3}\pi n) = \cos(\frac{2}{3}\pi n), & k=2 \\ \cos(\frac{30}{15}\pi n) = 1, & k=3 \\ 2\cos(\frac{2}{3}\pi n) + 1, & k=4 \end{cases}
\end{aligned}
$$

(d) The signals are

$$
\begin{aligned}
z_1(t) &= \cos(10\pi t) \\
z_2(t) &= \cos(10\pi t) \\
z_3(t) &= 1 \\
z_4(t) &= 1 + 2\cos(10\pi t)
\end{aligned}
$$

17. A continuous signal $x_a(t)$ has the Fourier Transform

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

such that $X_a(\Omega) = 0, |\Omega| > 2\pi \times 1$ rad/sec.

(a) Express the discrete time Fourier transform

$$X_b(\omega) = \sum_{n=-\infty}^{\infty} x_b(n) e^{-j\omega n}$$

in terms of $X_a(\Omega)$,where $x_b(n) = x_a(n + 0.25), \forall n \in Intergers$.

(b) Can the continuous time Fourier transform $X_a(\Omega)$ be uniquely determined from $X_b(W)$? If yes, how; if not, why not?

(c) Now assume that in addition to $x_b(n)$, another set of samples $x_c(n) = x_a(n)$ is obtained. Can $x_a(t)$ be uniquely determined from $x_b(n)$ and $x_c(n)$? If yes, how; if not, why not?

(d) What conclusions might you draw about sampling of bandlimited signals on the basis of your results?

Answer to 17 (a)

$$
\begin{aligned}
X_b(\omega) &= \sum_{n=-\infty}^{\infty} x_b(n) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\Omega) e^{j\Omega(n+0.25)} d\Omega e^{-j\omega n} \\
&= \int_{-\infty}^{\infty} X_a(\Omega) e^{j\Omega/4} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-jn(\omega-\Omega)} d\Omega \\
&= \int_{-\infty}^{\infty} X_a(\Omega) e^{j\Omega/4} \sum_{k=-\infty}^{\infty} \delta(\omega - \Omega + 2\pi k) d\Omega \\
&= \sum_{k=-\infty}^{\infty} X_a(\omega + 2\pi k) e^{j(\omega+2\pi k)/4}
\end{aligned}
$$

(b) $X_a(\Omega)$ can not be determined from $X_b(\omega)$ because $x_b(n)$ is undersampled.

(c)

$$X_c(\omega) = \sum_{k=-\infty}^{\infty} X_a(\omega + 2\pi k)$$

$X_a(\omega)$ can be obtained from $x_c(n)$ and $x_b(n)$.

Since $X_a(\omega)$ is bandlimited to $[-2\pi, 2\pi]$ rad/sec. For $0 \leq \omega < 2\pi$,

$$X_c(\omega) = X_a(\omega) + X_a(\omega - 2\pi)$$

While

$$X_b(\omega) = X_a(\omega)e^{j\omega/4} + X_a(\omega - 2\pi)e^{j\omega/4}e^{-j\pi/2}$$

Thus,

$$\begin{bmatrix} X_c(\omega) \\ X_b(\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{j\omega/4} & -je^{j\omega/4} \end{bmatrix} \begin{bmatrix} X_a(\omega) \\ X_a(\omega - 2\pi) \end{bmatrix}$$

So we can obtain $X_a(\omega)$ by a matrix inverse

$$\begin{bmatrix} X_a(\omega) \\ X_a(\omega - 2\pi)) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{j\omega/4} & -je^{j\omega/4} \end{bmatrix}^{-1} \begin{bmatrix} X_c(\omega) \\ X_b(\omega) \end{bmatrix}$$

(d) The implication is that a bandlimited signal can be reconstructed perfectly as long as the average sampling rate satisfies the Shannon-Nyquist Theorem.
However, if the samples are too closely spaced,
i.e. $x_b(n) = x_a(n + 1/N)$, then the matrix inverse becomes troublesome. The matrix

$$\begin{bmatrix} 1 & 1 \\ e^{j\omega/4} & -je^{j\omega/4} \end{bmatrix}^{-1}$$

is very difficult to compute for large N.

18. The autocorrelation sequence of a signal $X(n)$ is defined as

$$R_x(n) = \sum_{k=-\infty}^{\infty} x^*(k)x(n+k)$$

Express the Fourier transform of $R_x(n)$ in terms of $X(\omega)$, the Fourier transform of $x(n)$.

Answer to 18 Using the change of variable: $r = -k$, we can rewrite $R_x(n)$ as:

$$R_x(n) = \sum_{r=-\infty}^{\infty} x^*(-r)x(n-r) = x^*(-n) * x(n)$$

By the symmetry property of DTFT : $x^*(-n) \leftrightarrow X^*(\omega)$, thus

$$R_x(\omega) = X^*(\omega)X(\omega) = |X(\omega)|^2.$$