EECS 20. Final Exam Solution. December 21, 2004.

1. **15 points, 3 points each part** Give precise definitions of the following:

   (a) The space *Images* of all 600×900 images with pixel values in $\{0, 1, \cdots, 255\}$.
   Answer

   $$\textit{Images} = [\{1, \cdots, 600\} \times \{1, \cdots, 900\} \to \{0, \cdots, 255\}].$$

   (b) A one-to-one and onto function $f : [0, \infty) \to [0, 1)$.
   Answer The function f works:

   $$\forall x \in [0, \infty), \quad f(x) = 1 - e^{-x}.$$

   (c) A linear one-to-one function $f : \textit{Complex} \to \textit{Reals}^2$.
   Answer The function f works:

   $$\forall z \in \textit{Complex}, \quad f(z) = (\textit{Re}(z), \textit{Im}(z)).$$

   (d) The spaces *ContSignals* and *DiscSignals* of continuous-time and discrete-time complex-valued signals (use the [ ] notation):
   Answer We have

   $$\begin{aligned} \textit{ContSignals} &= [\textit{Reals} \to \textit{Complex}] \\ \textit{DiscSignals} &= [\textit{Ints} \to \textit{Complex}] \end{aligned}$$

   Now define the system

   $$\textit{Sampler}_T : \textit{ContSignals} \to \textit{DiscSignals}$$

   which samples its input every T sec.
   Answer

   $$\forall x \in \textit{ContSignals} \forall x \in \textit{Ints}, \textit{Sampler}_T(x)(n) = x(nT).$$

   (e) The convolution $z = x * y$ when

      i. $x, y : \textit{Reals} \to \textit{Reals}$.
      Answer
      $$\forall t \in \textit{Reals}, z(t) = \int_{-\infty}^{\infty} x(s)y(t-s)ds.$$

      ii. $x, y : \textit{Ints} \to \textit{Reals}$.
      Answer
      $$\forall n \in \textit{Ints}, z(n) = \sum_{m=-\infty}^{\infty} x(n-m)y(m).$$

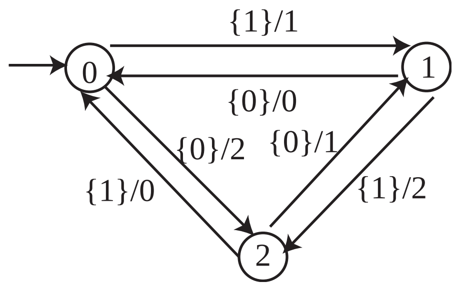
2. **15 points** Design two state machines, both with *Inputs* $= \{0, 1\}$, *Outputs* $= \{0, 1, 2\}$, and with input-output functions S_1, S_2 given below.

(a) **7 points**

$$\forall x, \forall n \quad S_1(x)(n) = (n_1 - n_0) \bmod 3,$$

in which n_1 and n_0 are the numbers of 1's and 0's in $(x(0), \cdots, x(n))$, respectively.

Answer The state machine below works:

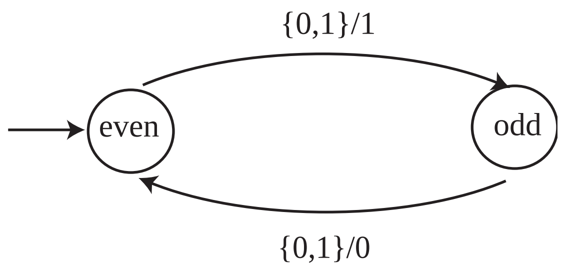


(b) **8 points**

$$\forall x, \forall n \quad S_2(x)(n) = \begin{cases} 0, & \text{if } (n_1 - n_0) \text{ is even} \\ 1, & \text{if } (n_1 - n_0) \text{ is odd} \end{cases}$$

in which n_1, n_0 are as above.

Answer The state machine below works:



3. **15 points, 5 points each part** Consider a LTI system with $[A, b, c, d]$ representation given by:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^T = [1 \;\; 0], \quad d = 0.$$

(a) Calculate the zero-input state response when the initial state is $s(0) = [s_1(0) \quad s_2(0)]^T$.
Answer The zero-input state response is $s(n) = A^n s(0), n \geq 0$. We have

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \; \cdots, A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

so

$$s(n) = [s_1(0) + ns_2(0) \quad s_2(0)]^T.$$

(b) Calculate the (zero-state) impulse response, h.
Answer The impulse response h is given by:

$$\begin{aligned} h(n) &= \begin{cases} 0, & n < 0 \\ d = 0, & n = 0 \\ c^T A^{n-1} b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = n-1, & n \geq 1. \end{cases} \\ &= \begin{cases} 0, & n \leq 0 \\ n-1, & n \geq 1 \end{cases} \end{aligned}$$

(c) Calculate the response $y(n), n \geq 0$ when the initial state is $s(0) = [1 \quad 1]^T$ and the input signal is $\forall n \geq 0, x(n) = \delta(n-1)$.
Answer The response is: $\forall n \geq 0$

$$\begin{aligned} y(n) &= c^T s(n) + h(n-1) \\ &= s_1(0) + ns_2(0) + h(n-1) \\ &= \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 2n-1, & n \geq 2 \end{cases} \end{aligned}$$

4. **20 points** Consider the difference equation

$$y(n) - 2y(n-1) = x(n) - 3x(n-1).$$

(a) **4 points** Take the state at time n as $s(n) = [y(n-1), x(n-1)]^T$ and write down the $[A, b, c, d]$ representation of the system.

Answer We have

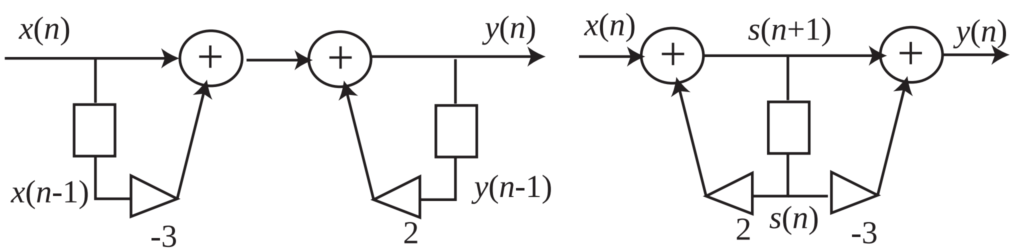
$$\begin{aligned} s(n+1) &= \begin{bmatrix} y(n) \\ x(n) \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(n) \\ y(n) &= [-2 \quad -3]\, s(n) + [1]x(n), \end{aligned}$$

from which

$$A = \begin{bmatrix} -2 & -3 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c^T = [-2 \quad -3], \quad d = 1.$$

(b) **4 points** Implement the difference equation using two delay elements whose outputs are the two state components.

Answer The implementation is drawn on the left:



(c) **6 points** Find another implementation using only *one* delay element and find the $[A, b, c, d]$ representation for this implementation.

Answer The implementation is on the right. The representation is

$$\begin{aligned} s(n+1) &= 2s(n+1) + x(n) \\ y(n) &= -3s(n) + s(n+1) \\ &= -s(n) + x(n) \end{aligned}$$

so,

$$A = 2, \quad b = 1, \quad c = -1, \quad d = 1$$

(d) **6 points** Determine the zero-state impulse response.

Answer We know that

$$h(n) = \begin{cases} d = 1, & n = 0 \\ cA^{n-1}b = -2^{n-1}, & n \geq 1 \end{cases}$$

5. **15 points** The bandwidth of a continuous time signal x with FT X is by definition the smallest frequency ω_B such that $X(\omega) = 0$ for $|\omega| > \omega_B$.

(a) **3 points** What is the bandwidth of the signals: $\forall t \in$ *Reals*,

$$x_k(t) = \sin(10k\pi t), k = 1, 2, 3; \quad x_4(t) = x_1(t) + x_2(t) + x_3(t).$$

Specify the units of the bandwidth.

Answer The bandwidth of x_k is $10k\pi$ rad/sec, for $k = 1, 2, 3$. The bandwidth of x_4 is $30k\pi$ rad/sec.

(b) **3 points** What is the FT of $x_k, k = 1, \cdots, 4$?

Answer The FT is: $\forall \omega \in$ *Reals*

$$\begin{aligned} X_k(\omega) &= -i\pi[\delta(\omega - 10k\pi) - \delta(\omega + 10k\pi)], k = 1, 2, 3 \\ X_4(\omega) &= -i\pi \sum_{k=1}^{3} [\delta(\omega - 10k\pi) - \delta(\omega + 10k\pi)] \end{aligned}$$

(c) **4 points** Suppose x_k is sampled at frequency $\omega_s = 30\pi$ rad/sec. Find a simple expression for the sampled signal y_k.

Answer The sampling time is $T = 2\pi/\omega_s = 1/15$ s. So

$$\begin{aligned} \forall n \in \textit{Ints}, \quad y_k(n) &= x_k(nT) \\ &= \sin(\frac{2}{3}k\pi n), k = 1, 2, 3. \end{aligned}$$

Since $\sin(4/3\pi n) = -\sin(2/3\pi n)$ and $\sin(6/3\pi n) = 0$,

$$\begin{aligned} \forall n \in \textit{Ints}, \quad y_1(n) &= \sin(\frac{2}{3}k\pi n), \\ y_2(n) &= -\sin(\frac{2}{3}\pi n), \\ y_3(n) &= 0, \\ y_4(n) &= y_1(n) + y_2(n) + y_3(n) = 0 \end{aligned}$$

(d) **5 points** Find signals z_k : *Reals* $\to$ *Reals* such that (i) the bandwidth of z_k is smaller than 15π rad/sec, which is one-half the sampling frequency; and (ii) if z_k is sampled at frequency ω_s it also yields the signal y_k.

Answer From part (c), we get:

$$\begin{aligned} \forall t \in \textit{Reals}, \quad z_1(t) &= \sin(10\pi t) \\ z_2(t) &= -\sin(10\pi t) \\ z_3(t) = z_4(t) &= 0 \end{aligned}$$

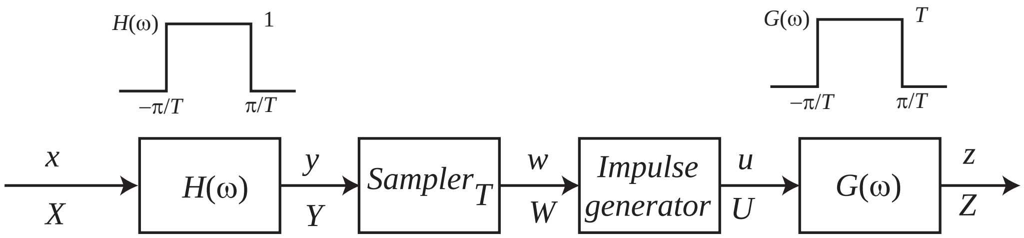


Figure 1: Setup for problem 6

6. **20 points, 5 points each part** Consider the setup of figure 1. The filters H, G are as shown; the sampling period is T seconds.

   (a) Express w, u in terms of y and W, U in terms of Y. **Answer** We have

   $$\forall n, w(n) = y(nT) \text{ and } \forall t, u(t) = \sum_{-\infty}^{\infty} y(nT)\delta(t - nT),$$

   and

   $$\forall \omega, W(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y(\frac{\omega - 2\pi k}{T}) \text{ and } \forall \omega, U(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y(\omega - \frac{2\pi k}{T})$$

   (b) Express Z in terms of X.
   Answer By Shannon-Nyquist theorem,

   $$\forall \omega, \quad Z(\omega) = Y(\omega) = \begin{cases} X(\omega), & |\omega| \le \frac{\pi}{T} \\ 0, & |\omega| > \frac{\pi}{T} \end{cases}$$

   (c) Determine y and z for $T = 0.1s$ and $\forall t, x(t) = \sin(25\pi t) + \sin(5\pi t)$.
   Answer The higher frequency term in x will be eliminated by H. So,

   $$\forall t, \quad y(t) = z(t) = \sin(5\pi t).$$

   (d) Suppose in this setup H is changed to $\forall \omega, H(\omega) = 1$. Take T, x as above, and determine z.
   Answer The higher frequency term will be aliased by the sampling as $\sin(25\pi t - 2\pi\omega_s t) = \sin(5\pi t)$. Hence

   $$\forall t, \quad z(t) = 2\sin(5\pi t).$$

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