

## EECS 20. Midterm No. 2 Practice Problems, November 10, 2004.

1. When the inputs to a time-invariant system are:  $\forall n$ ,

$$\begin{aligned}x_1(n) &= 2\delta(n-2) \\x_2(n) &= \delta(n+1)\end{aligned}, \quad \text{where } \delta \text{ is the Kronecker delta}$$

the corresponding outputs are

$$\begin{aligned}y_1(n) &= \delta(n-2) + 2\delta(n-3) \\y_2(n) &= 2\delta(n+1) + \delta(n)\end{aligned}, \quad \text{respectively.}$$

Is this system is linear? Give a proof or a counter-example.

2. Consider discrete-time systems with input and output signals  $x, y \in [\text{Integers} \rightarrow \text{Reals}]$ . Each of the following relations defines such a system. For each, indicate whether it is linear(L), time-invariant (TI), both(LTI), or neither (N). Give a proof or counter-example.

(a)  $y(n) = g(n)x(n)$

(b)  $y(n) = e^{x(n)}$

3. (a) An LTI system with input signal  $x$  and output signal  $y$  is described by the differential equation

$$\ddot{y}(t) + 2\dot{y}(t) + 0.5y(t) = x(t).$$

Suppose the input signal is  $\forall t, x(t) = e^{i\omega t}$ , where  $\omega$  is fixed. What is its output signal  $y$ ?

- (b) Another LTI system is subject to the differential equation

$$\ddot{y}(t) + y(t) = \dot{x}(t) + x(t)$$

i. What is the frequency response?

ii. What is the magnitude and phase of the frequency response for  $\omega = 0.5$ ?

4. For this problem, assume discrete time everywhere. Given two LTI systems  $S$  and  $T$  suppose signal  $f$  is input into  $S$  and  $g$  into  $T$ . The input and output signals are displayed in figure 1. Are the two systems identical, that is,  $S = T$ ?

5. A system is described by the difference equation

$$y(n) = x(n) + bx(n-1) + ay(n-1),$$

wherein  $a, b$  are constants.

- (a) Obtain the  $[A, b, c^T, d]$  representation of this system by:

i. choosing the state,

ii. calculating  $A, b, c^T, d$  for your choice of state.

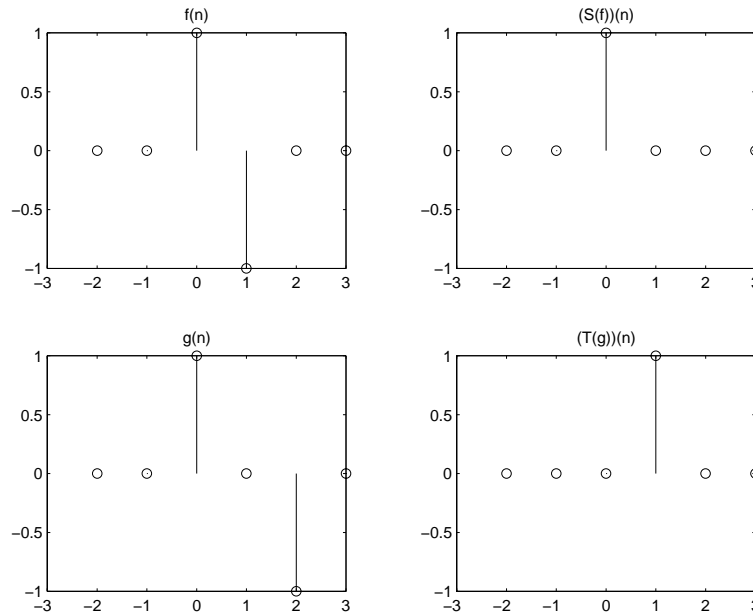


Figure 1: Signals for Problem 4

- (b) If  $x(n - 1) = 0, y(n - 1) = 1$ , calculate the zero-input (i.e.  $x(n) = 0, n \geq 0$ ) state response.
- (c) Calculate the frequency response of this system.

6. For the linear difference equation

$$y(n) = 0.5y(n - 1) + x(n),$$

- (a) Taking the state at time  $n$  to be  $s(n) = y(n - 1)$ , write down the zero-input response, the zero-state impulse response  $h : \text{Ints} \rightarrow \text{Reals}$ , the zero-state response, and the (full) response.
- (b) Show that the zero-input response  $y_{zi}$  is a linear function of the initial state, i.e. it is of the form

$$\forall n \geq 0, \quad y_{zi}(n) = a(n)s(0),$$

for some constant coefficients  $a(n)$ . Then show that

$$\lim_{n \rightarrow \infty} y_{zi}(n) = 0$$

- (c) Suppose  $s_0$  is the initial state and the input is a unit step, i.e.  $x(n) = 1, n \geq 0; = 0, n < 0$ . Determine the response  $y(n), n \geq 0$ , and calculate the steady state response

$$y_{ss} = \lim_{n \rightarrow \infty} y(n).$$

- (d) Plot the input, output and the steady state value in the previous part.
- (e) Calculate the frequency response  $H : \text{Reals} \rightarrow \text{Complex}$  and plot the magnitude and phase response.

(f) Suppose  $x(n) = 1, -\infty < n < \infty$ . What is the output  $y(n), -\infty < n < \infty$  and compare it with  $y_{ss}$ .

7. Suppose  $x$  is a continuous-time periodic signal, with period  $p$  and exponential FS representation,

$$\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(ik\omega_0 t),$$

in which  $\omega_0 = 2\pi/p$ .

(a) Write down the formula for  $X_k$  in terms of  $x$ .

(b) Consider the signal  $y$ ,

$$\forall t, \quad y(t) = x(\alpha t),$$

in which  $\alpha > 0$  is some positive constant.

i. Show that  $y$  is periodic and find its period  $q$ .

ii. Suppose  $y$  has FS representation

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k \exp(k\omega_1 t),$$

What is  $\omega_1$ ? Determine the  $Y_k$  in terms of the  $X_k$ .

8. Give an example of a nonlinear, time-invariant system  $S$  that is **not** memoryless. Time is discrete.

(a) Show that  $S$  is nonlinear, time-invariant, and not memoryless.

(b) Suppose  $x : \text{Ints} \rightarrow \text{Reals}$  is periodic with period  $p$ . Let  $y = S(x)$ . Is  $y$  periodic?

(c) Suppose  $Q$  is another discrete-time, time-invariant system. Is the cascade composition  $S \circ Q$  time-invariant? Give a proof or a counterexample.

(d) Define the system  $R$  by reversing time:  $\forall x, n, R(x)(n) = S(x)(-n)$ . Is  $R$  time-invariant? Why? If  $x$  is periodic as above and  $w = R(x)$ , is  $w$  periodic? Why.

9. You are given three kinds of building blocks for discrete-time systems: one-unit delay; gains; and adders.

(a) Use these building blocks to implement the system:

$$y(n) = 0.5y(n-2) + x(n) + x(n-1).$$

(b) Take the outputs of the delay elements as the state and give a  $[A, b, c^T, d]$  representation of this system.

(c) You are allowed to set the output of the delay elements to any value at time  $n = 0$ . Select these values so that the output of your implementation is the solution  $y(n), n \geq 0$  for any input  $x(n), n \geq 0$  and initial conditions:  $y(-1) = 0.5, y(-2) = 0.8, x(-1) = 1$ . Now suppose  $x(0) = x(1) = x(2) = 0$ . Calculate  $y(0), y(1), y(2)$ .

10. An integrator can be used as a building block: For any input  $x : \text{Reals}_+ \rightarrow \text{Reals}$ , its output is:

$$\forall t \geq 0, \quad y(t) = y_0 + \int_0^t x(s) ds.$$

The ‘initial condition’  $y(0)$  can be set.

Use integrators, gains and adders to implement the system:

$$\frac{d^2 y}{dt^2}(t) - y(t) = x(t),$$

with initial condition  $y(0) = 1, \dot{y}(0) = 0.4$ .

**Hint** First convert a differential equation into an integral equation and then implement.

11. A periodic signal  $x : \text{Reals} \rightarrow \text{Reals}$  is given by

$$\forall t, \quad x(t) = [1 + \cos(2\pi \times 10t)] \times \cos(2\pi \times 400t).$$

- (a) What are the fundamental frequency  $\omega_0$  and period  $T_0$  of  $x$ ? Calculate the Fourier Series of  $x$  in the forms:

$$\begin{aligned} \forall t, \quad x(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) \\ &= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \end{aligned}$$

Is  $X_k = X_{-k}^*$ ?

- (b) Suppose the LTI system  $S$  has frequency response

$$\forall \omega, \quad H(\omega) = \begin{cases} 1, & \text{if } 2\pi \times 395 \leq |\omega| \leq 2\pi \times 405 \\ 0, & \text{otherwise} \end{cases}$$

Plot the magnitude and phase response of  $H$ . Repeat part 11a for  $y$ .

12. Give the ABCD state space representation of a discrete-time system with frequency response  $H(\omega)$ , where:

$$H(\omega) = \frac{2 + e^{-j\omega}}{1 - 3e^{-3j\omega}}$$

**Hint:** First find a difference equation which has the given frequency response. Then find the state space representation.

13. You are given the signal  $\forall t x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$  to use as input to a system with frequency response  $H(\omega) = |\omega|$ . Answer the following questions based on this setup.

- (a) Indicate the Fourier series expansion (in cosine format) of  $x$  by writing the nonzero values of  $A_0, A_k$ , and  $\phi_k$  in the expansion  $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$ .

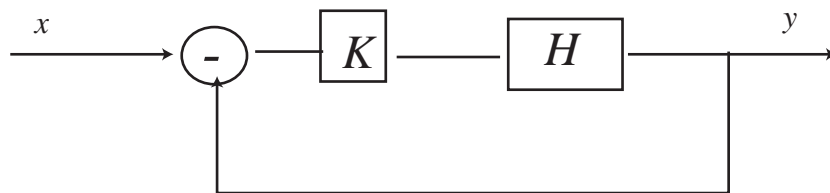


Figure 2: Feedback system for problem 14

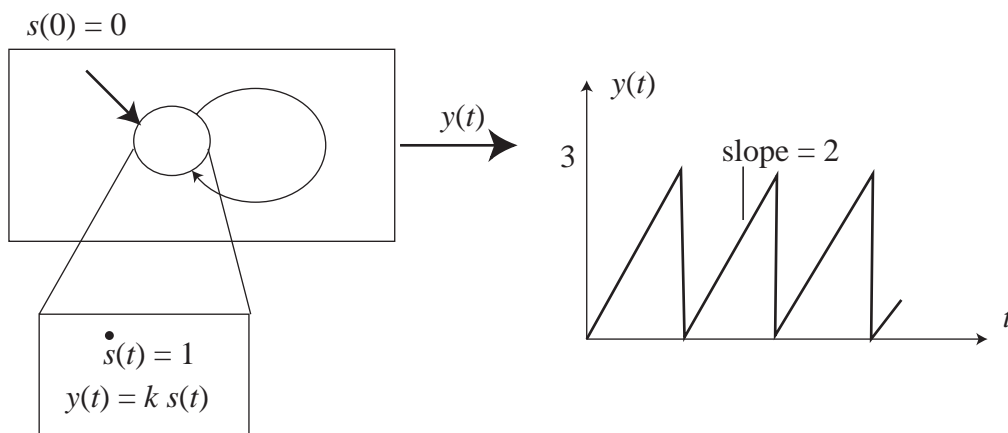


Figure 3: Hybrid system for problem 15

- (b) Indicate the Fourier series expansion (in complex exponential format) of  $x(t)$  by writing the nonzero values of the complex coefficients  $X_k$  in the expansion  $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$ .
- (c) Give  $y$ , the output of the system with input  $x$ .
14. In the negative feedback system of figure 2 assume that  $H(\omega) = [1 + i\omega]^{-1}$ . Let  $G$  be the closed-loop frequency response. For  $K = 1, 10, 100$
- Plot the magnitude and phase response of  $G$ ; and
  - determine the bandwidth  $\omega$  at which  $\angle G(\omega) = \pi/4$ .
15. Determine the ‘gain’  $k$  and the guard so that the output of the hybrid system is as shown in figure 3.