EECS 20. Midterm No. 2 Practice Problems, November 10, 2004.

1. When the inputs to a time-invariant system are: $\forall n$,

$$\begin{array}{rcl} x_1(n) & = & 2\delta(n-2) \\ x_2(n) & = & \delta(n+1) \end{array}, \quad \text{where } \delta \text{ is the Kronecker delta}$$

   the corresponding outputs are

$$\begin{array}{rcl} y_1(n) & = & \delta(n-2) + 2\delta(n-3) \\ y_2(n) & = & 2\delta(n+1) + \delta(n) \end{array}, \quad \text{respectively.}$$

   Is this system is linear? Give a proof or a counter-example.

2. Consider discrete-time systems with input and output signals $x, y \in [Integers \to Reals]$. Each of the following relations defines such a system. For each, indicate whether it is linear(L), time-invariant (TI), both(LTI), or neither (N). Give a proof or counter-example.

   (a) $y(n) = g(n)x(n)$

   (b) $y(n) = e^{x(n)}$

3. (a) An LTI system with input signal x and output signal y is described by the differential equation

$$\ddot{y}(t) + 2\dot{y}(t) + 0.5y(t) = x(t).$$

   Suppose the input signal is $\forall t, x(t) = e^{i\omega t}$, where ω is fixed. What is its output signal y?

   (b) Another LTI system is subject to the differential equation

$$\ddot{y}(t) + y(t) = \dot{x}(t) + x(t)$$

   i. What is the frequency response?

   ii. What is the magnitude and phase of the frequncy response for $\omega = 0.5$?

4. For this problem, assume discrete time everywhere. Given two LTI systems S and T suppose signal f is input into S and g into T. The input and output signals are displayed in figure1. Are the two systems identical, that is, $S = T$?

5. A system is described by the difference equation

$$y(n) = x(n) + bx(n-1) + ay(n-1),$$

   wherein a, b are constants.

   (a) Obtain the $[A, b, c^T, d]$ representation of this system by:

   i. choosing the state,

   ii. calculating A, b, c^T, d for your choice of state.

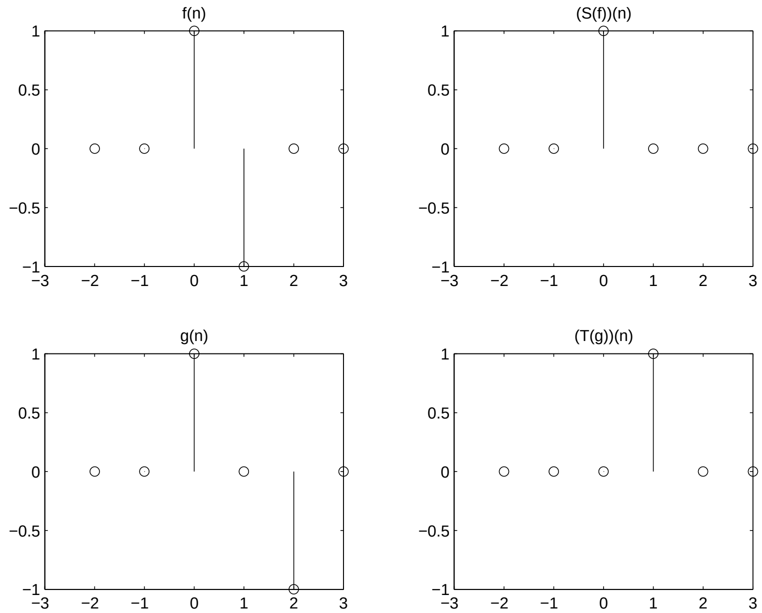

Figure 1: Signals for Problem 4

(b) If $x(n-1) = 0, y(n-1) = 1$, calculate the zero-input (i.e. $x(n) = 0, n \geq 0$) state response.

(c) Calculate the frequency response of this system.

6. For the linear difference equation

$$y(n) = 0.5y(n-1) + x(n),$$

(a) Taking the state at time n to be $s(n) = y(n-1)$, write down the zero-input response, the zero-state impulse response h : *Ints* $\rightarrow$ $Reals$, the zero-state response, and the (full) response.

(b) Show that the zero-input response y_{zi} is a linear function of the initial state, i.e. it is of the form

$$\forall n \geq 0, \quad y_{zi}(n) = a(n)s(0),$$

for some constant coefficients $a(n)$. Then show that

$$\lim_{n\to\infty} y_{zi}(n) = 0$$

(c) Suppose s_0 is the initial state and the input is a unit step, i.e. $x(n) = 1, n \geq 0; = 0, n < 0$. Determine the response $y(n), n \geq 0$, and calculate the steady state response

$$y_{ss} = \lim_{n\to\infty} y(n).$$

(d) Plot the input, output and the steady state value in the previous part.

(e) Calculate the frequency response H : *Reals* $\rightarrow$ *Complex* and plot the magnitude and phase response.

(f) Suppose $x(n) = 1, -\infty < n < \infty$. What is the output $y(n), -\infty < n < \infty$ and compare it with y_{ss}.

7. Suppose x is a continuous-time periodic signal, with period p and exponential FS representation,

$$\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(ik\omega_0 t),$$

in which $\omega_0 = 2\pi/p$.

(a) Write down the formula for X_k in terms of x.

(b) Consider the signal y,

$$\forall t, \quad y(t) = x(\alpha t),$$

in which $\alpha > 0$ is some positive constant.

i. Show that y is periodic and find its period q.

ii. Suppose y has FS representation

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k \exp(k\omega_1 t),$$

What is ω_1? Determine the Y_k in terms of the X_k.

8. Give an example of a nonlinear, time-invariant system S that is **not** memoryless. Time is discrete.

(a) Show that S is nonlinear, time-invariant, and not memoryless.

(b) Suppose x : *Ints* $\to$ *Reals* is periodic with period p. Let $y = S(x)$. Is y periodic?

(c) Suppose Q is another discrete-time, time-invariant system. Is the cascade composition $S \circ Q$ time-invariant? Give a proof or a counterexample.

(d) Define the system R by reversing time: $\forall x, n, R(x)(n) = S(x)(-n)$. Is R time-invariant? Why? If x is periodic as above and $w = R(x)$, is w periodic? Why.

9. You are given three kinds of building blocks for discrete-time systems: one-unit delay; gains; and adders.

(a) Use these building blocks to implement the system:

$$y(n) = 0.5y(n-2) + x(n) + x(n-1).$$

(b) Take the outputs of the delay elements as the state and give a $[A, b, c^T, d]$ representation of this system.

(c) You are allowed to set the output of the delay elements to any value at time $n = 0$. Select these values so that the output of your implementation is the solution $y(n), n \geq 0$ for any input $x(n), n \geq 0$ and initial conditions: $y(-1) = 0.5, y(-2) = 0.8, x(-1) = 1$. Now suppose $x(0) = x(1) = x(2) = 0$. Calculate $y(0), y(1), y(2)$.

10. An integrator can be used as a building block: For any input x : *Reals*$_+ \to$ *Reals*, its output is:

$$\forall t \geq 0, \quad y(t) = y_0 + \int_0^t x(s)ds.$$

The 'initial condition' $y(0)$ can be set.

Use integrators, gains and adders to implement the system:

$$\frac{d^2y}{dt^2}(t) - y(t) = x(t),$$

with iniital condition $y(0) = 1, \dot{y}(0) = 0.4$.
Hint First convert a differential equation into an integral equation and then implement.

11. A periodic signal x : *Reals* $\to$ *Reals* is given by

$$\forall t, \quad x(t) = [1 + \cos(2\pi \times 10t)] \times \cos(2\pi \times 400t).$$

   (a) What are the fundamental frequency ω_0 and period T_0 of x? Calculate the Fourier Series of x in the forms:

$$\begin{aligned} \forall t, \quad x(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) \\ &= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \end{aligned}$$

   Is $X_k = X^*_{-k}$?

   (b) Suppose the LTI system S has frequency response

$$\forall \omega, \quad H(\omega) = \begin{cases} 1, & \text{if } 2\pi \times 395 \leq |\omega| \leq 2\pi \times 405 \\ 0, & \text{otherwise} \end{cases}$$

   Plot the magnitude and phase response of H. Repeat part 11a for y.

12. Give the ABCD state space representation of a discrete-time system with frequency response $H(\omega)$, where:

$$H(\omega) = \frac{2 + e^{-j\omega}}{1 - 3e^{-3jw}}$$

Hint: First find a difference equation which has the given frequency response. Then find the state space representation.

13. You are given the signal $\forall t x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$ to use as input to a system with frequency response $H(\omega) = |\omega|$. Answer the following questions based on this setup.

   (a) Indicate the Fourier series expansion (in cosine format) of x by writing the nonzero values of A_0, A_k, and ϕ_k in the expansion $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$.

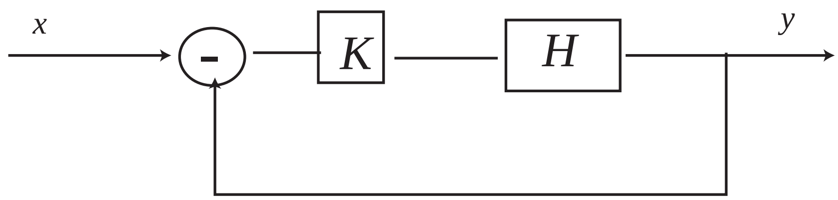

Figure 2: Feedback system for problem 14

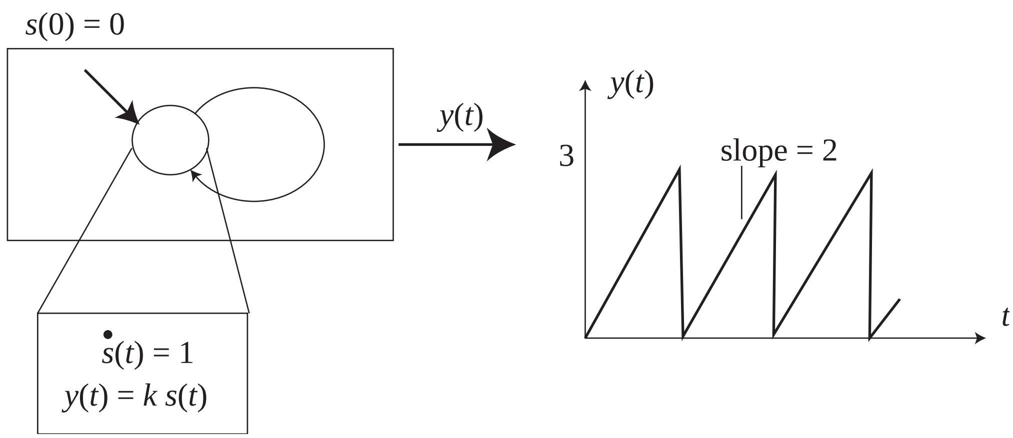

Figure 3: Hybrid system for problem 15

(b) Indicate the Fourier series expansion (in complex exponential format) of $x(t)$ by writing the nonzero values of the complex coefficients X_k in the expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$.

(c) Give y, the output of the system with input x.

14. In the negative feedback system of figure 2 assume that $H(\omega) = [1 + i\omega]^{-1}$. Let G be the closed-loop frequency response. For $K = 1, 10, 100$

    (a) Plot the magnitude and phase response of G; and

    (b) determine the bandwidth ω at which $\angle G(\omega) = \pi/4$.

15. Determine the 'gain' k and the guard so that the output of the hybrid system is as shown in figure 3.

16. Suppose we have a signal $x : Integers \to Reals$, which is zero for all negative time, that is,

$$\forall k < 0, x(k) = 0.$$

Suppose a signal $y : Integers \to Reals$ is obtained by filtering x as in Figure 4, with the following result:

$$\forall k < 0, \qquad y(k) = 0$$

$$\begin{aligned} \text{for } k = 0, \quad & y(k) = x(0) \\ \forall k > 0, \quad & y(k) = x(k-1) + x(k) \end{aligned}$$

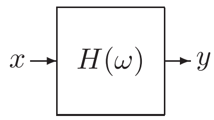

Figure 4: The filtering system.

(a) Find the impulse response h of the system in Figure 4 and the frequency response H.

(b) Suppose we receive the signal y, and we wish to recover the signal x. We can use a feedback connection to achieve this result. Design the impulse response g and frequency response G of the system used in feedback in Figure 5 so that the feedback system recovers the signal x.

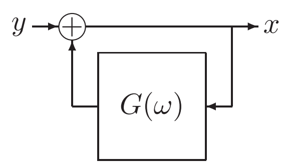

Figure 5: The feedback system.

(c) Find the impulse response f and the frequency response $F(\omega)$ of the overall feedback system from y to x in Figure 5.

17. Suppose that we have a SISO continuous time system of the following form:

$$\begin{aligned} \dot{s}(t) &= As(t) + bx(t), \\ y(t) &= c^T s(t). \end{aligned}$$

We decide to define a new state function $\tilde{s} : Reals \to Reals^N$, where

$$\forall t \in Reals, \tilde{s}(t) = Ts(t),$$

and T is an invertible $N \times N$ matrix. Find $\tilde{A}, \tilde{b}, \tilde{c}^T$ such that

$$\begin{aligned} \dot{\tilde{s}}(t) &= \tilde{A}\tilde{s}(t) + \tilde{b}x(t), \\ y(t) &= \tilde{c}^T \tilde{s}(t). \end{aligned}$$

In this case, we have transformed the state, but we still maintain the same input/output behavior.