EECS 20. Midterm No. 2 Practice Problems Solution, November 10, 2004.

1. When the inputs to a time-invariant system are: $\forall n$,

$$\begin{array}{rcl} x_1(n) & = & 2\delta(n-2) \\ x_2(n) & = & \delta(n+1) \end{array}, \quad \text{where } \delta \text{ is the Kronecker delta}$$

   the corresponding outputs are

$$\begin{array}{rcl} y_1(n) & = & \delta(n-2) + 2\delta(n-3) \\ y_2(n) & = & 2\delta(n+1) + \delta(n) \end{array}, \quad \text{respectively.}$$

   Is this system is linear? Give a proof or a counter-example.

   Answer to 1 The system is not linear. From time-invariance we see that for the second pair of input and output,

$$\begin{array}{rcl} x_2(n-3) & = & \delta(n-2) \\ y_2(n-3) & = & 2\delta(n-2) + \delta(n-3) \end{array}$$

   So we can rewrite the first pair of input and output as

$$\begin{array}{rcl} x_1(n) & = & 2\delta(n-2) \\ & = & 2x_2(n-3) \\ y_1(n) & = & \delta(n-2) + 2\delta(n-3) \\ & \neq & 2y_2(n-3) = 4\delta(n-2) + 2\delta(n-3) \end{array}$$

   Therefore, the system is not linear.

2. Consider discrete-time systems with input and output signals $x, y \in [Integers \to Reals]$. Each of the following relations defines such a system. For each, indicate whether it is linear(L), time-invariant (TI), both(LTI), or neither (N). Give a proof or counter-example.

   (a) $y(n) = g(n)x(n)$

   (b) $y(n) = e^{x(n)}$

   Answer to 2
   (a) The system is linear:

$$\begin{array}{rcl} \hat{x}(n) & = & ax_1(n) + bx_2(n) \\ \hat{y}(n) & = & g(n)(ax_1(n) + bx_2(n)) \\ & = & ay_1(n) + by_2(n) \end{array}$$

Also the system is time-varying if g is not constant (so there exist n, n_0 so that $g(n) \neq g(n-n_0)$):

$$\begin{aligned}
\hat{x}(n) &= x(n-n_0) \\
\hat{y}(n) &= g(n)\hat{x}(n) \\
&= g(n)x(n-n_0) \\
&\neq y(n-n_0) \\
&= g(n-n_0)x(n-n_0)
\end{aligned}$$

(b) The system is non-linear:

$$\begin{aligned}
\hat{x}(n) &= ax_1(n) + bx_2(n) \\
\hat{y}(n) &= e^{\hat{x}(n)} \\
&= e^{ax_1(n)+bx_2(n)} \\
&= (y_1(n))^a (y_2(n))^b \\
&\neq ay_1(n) + by_2(n)
\end{aligned}$$

But the system is time-invariant:

$$\begin{aligned}
\hat{x}(n) &= x(n-n_0) \\
\hat{y}(n) &= e^{\hat{x}(n)} \\
&= e^{x(n-n_0)} \\
&= y(n-n_0)
\end{aligned}$$

3. (a) An LTI system with input signal x and output signal y is described by the differential equation

$$\ddot{y}(t) + 2\dot{y}(t) + 0.5y(t) = x(t).$$

Suppose the input signal is $\forall t, x(t) = e^{i\omega t}$, where ω is fixed. What is its output signal y?

(b) Another LTI system is subject to the differential equation

$$\ddot{y}(t) + y(t) = \dot{x}(t) + x(t)$$

  i. What is the frequency response?
  ii. What is the magnitude and phase of the frequncy response for $\omega = 0.5$?

Answer to 3

(a) The output signal is $\forall t, y(t) = H(\omega)e^{i\omega t}$. It follows that

$$-\omega^2 H(\omega)e^{i\omega t} + 2i\omega H(\omega)e^{i\omega t} + 0.5H(\omega)e^{i\omega t} = e^{i\omega t},$$

thus $H(\omega) = \frac{1}{-\omega^2+2i\omega+0.5}$, Hence

$$\forall t, y(t) = \frac{1}{-\omega^2 + 2i\omega + 0.5}e^{i\omega t}$$

(b) (i) The frequency response is $H(\omega) = \frac{i\omega+1}{-\omega^2+1}$.

(ii) Hence

$$|H(0.5)| = |\frac{4}{3} + i\frac{2}{3}| = \frac{2\sqrt{5}}{3}, \quad \angle H(0.5) = \tan^{-1} 0.5$$

4. For this problem, assume discrete time everywhere. Given two LTI systems S and T suppose signal f is input into S and g into T. The input and output signals are displayed in figure1. Are the two systems identical, that is, $S = T$?

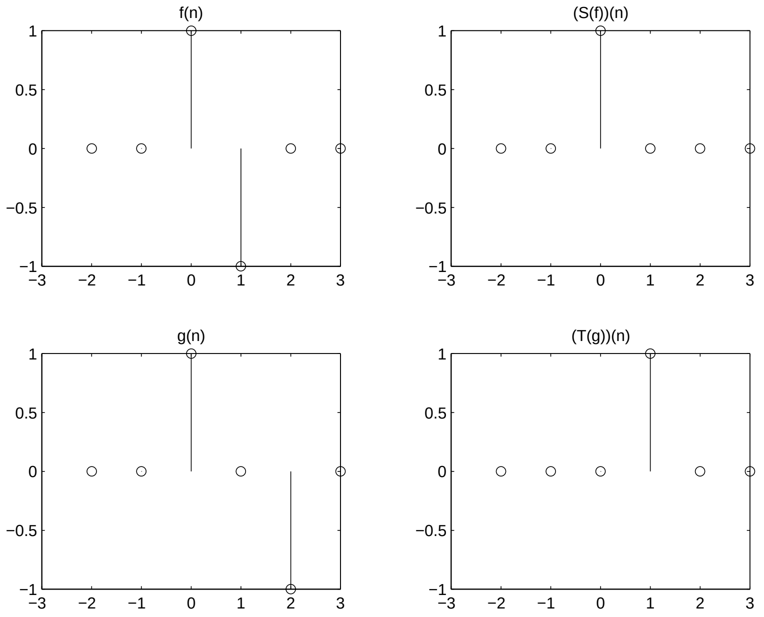


Figure 1: Signals for problem 4

Answer to 4 No. $S \neq T$ Argue by contradiction. Assume $S = T = R$, say. Observe that $f(n)$ is $(g - f)(n + 1)$. The figure below plots $R(g - f)(n) = T(g)(n) - S(f)(n)$ and $R((g - f))(n + 1) = R(f)(n) = S(f)(n)$. But the second plot is not the first plot delayed by 1.

5. A system is described by the difference equation

$$y(n) = x(n) + bx(n-1) + ay(n-1), \tag{1}$$

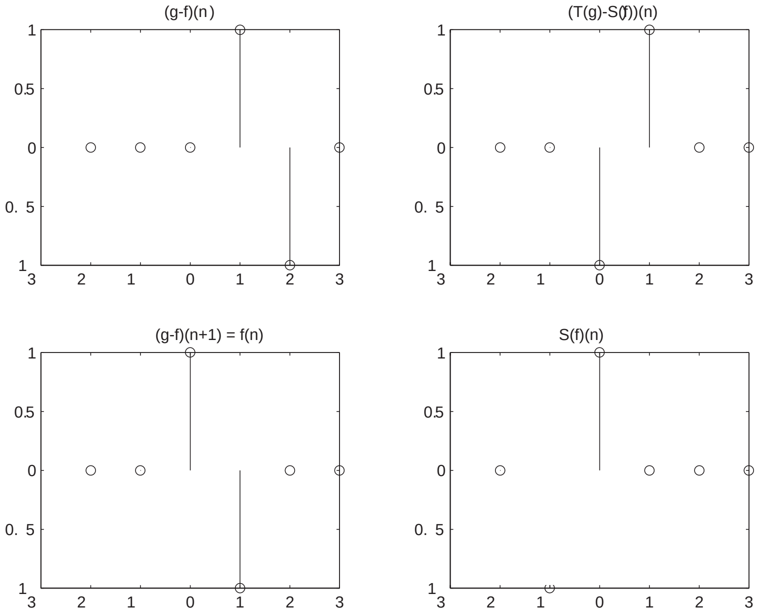
wherein a, b are constants.

(a) Obtain the $[A, b, c^T, d]$ representation of this system by:

   i. choosing the state,

   ii. calculating A, b, c^T, d for your choice of state.

(b) If $x(n-1) = 0, y(n-1) = 1$, calculate the zero-input (i.e. $x(n) = 0, n \geq 0$) state response.



(c) Calculate the frequency response of this system.

Answer to 5 (a) (i) Take the state as $s(n) = [x(n-1), y(n-1)]^T$.

(ii) Writing $s(n+1) = As(n) + bx(n)$ in expanded form gives

$$\begin{aligned} s(n+1) &= \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} x(n) \\ x(n) + bx(n-1) + ay(n-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ b & a \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(n), \end{aligned}$$

from which

$$A = \begin{bmatrix} 0 & 0 \\ b & a \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2}$$

and, since

$$y = [b \quad a] \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} + x(n),$$

so $c^T = [b \quad a], d = 1$.

(b) The zero-input state response is $s(n) = A^n s(0), n \geq 0$. So we need to calculate A^n, with A given in (2). By induction,

$$A^n = \begin{bmatrix} 0 & 0 \\ a^{n-1}b & a^n \end{bmatrix}$$

and since $s(0) = [0 \quad 1]^T$, $s(n) = [0 \quad a^n]$.

(c) To obtain the frequency response, substitute $x(n) = e^{jomegan}, y(n) = H(\omega)e^{i\omega n}$ in)(1) and simplify to get

$$\forall \omega, \quad H(\omega) = \frac{1 + be^{-i\omega}}{1 - ae^{-i\omega}}.$$

6. For the linear difference equation

$$y(n) = 0.5y(n-1) + x(n),$$

   (a) Taking the state at time n to be $s(n) = y(n-1)$, write down the zero-input response, the zero-state impulse response h : *Ints* $\to Reals$, the zero-state response, and the (full) response.

   (b) Show that the zero-input response y_{zi} is a linear function of the initial state, i.e. it is of the form

$$\forall n \geq 0, \quad y_{zi}(n) = a(n)s(0),$$

   for some constant coefficients $a(n)$. Then show that

$$\lim_{n\to\infty} y_{zi}(n) = 0$$

   (c) Suppose s_0 is the initial state and the input is a unit step, i.e. $x(n) = 1, n \geq 0; = 0, n < 0$. Determine the response $y(n), n \geq 0$, and calculate the steady state response

$$y_{ss} = \lim_{n\to\infty} y(n).$$

   (d) Plot the input, output and the steady state value in the previous part.

   (e) Calculate the frequency response H : *Reals* $\to$ *Complex* and plot the magnitude and phase response.

   (f) Suppose $x(n) = 1, -\infty < n < \infty$. What is the output $y(n), -\infty < n < \infty$ and compare it with y_{ss}.

Answer to 6 (a) The a, b, c, d representation is (with $s(n) = y(n-1)$)

$$s(n+1) = 0.5s(n) + x(n), \quad y(n) = 0.5s(n) + x(n).$$

The zero-input response ($x(n) = 0, n \geq 0$) is

$$s_{zi}(n) = 0.5^n s(0), \quad y_{zi}(n) = 0.5^{n+1}s(0) = 0.5^{n+1}y(-1). \tag{3}$$

The zero-state impulse response is

$$\forall n \geq 0, \quad h(n) = \begin{cases} d = 1, & n = 0 \\ ca^{n-1}b = 0.5^n, & n \geq 1 \end{cases} = 0.5^n.$$

So the full response is

$$y(n) = 0.5^{n+1}y(-1) + \sum_{m=0}^{n} 0.5^{n-m}x(m), n \geq 0. \tag{4}$$

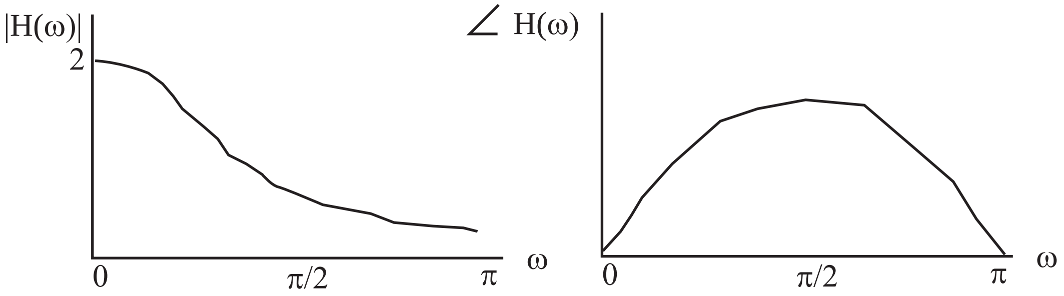


Figure 2: Plots for problem 6

(b) From (3) y_{zi} is a linear (time-varying) function of the initial state with $a(n) = 0.5^{n+1}$. Clearly, $y_{zi}(n) \to 0$ as $n \to \infty$.

(c)In (4) take $x(m) = 1, m \geq 0$ to get

$$\begin{aligned} y(n) &= 0.5^{n+1} s_0 + \sum_{m=0}^{n} 0.5^{n-m} \times 1 \\ &= 0.5^{n+1} s_0 + \sum_{k=0}^{n} 0.5^{k} = 0.5^{n+1} s_0 + \frac{1 - 0.5^{n+1}}{1 - 0.5} \\ \to y_{ss} = 2 \text{ as } n \to \infty \end{aligned}$$

(d) The plots are straightforward.

(e) The frequency response is

$$\forall \omega, \quad H(\omega) = \frac{1}{1 - 0.5e^{-i\omega}},$$

the magnitude response is

$$\forall \omega, \quad |H(\omega)| = \frac{1}{[1.25 - \cos(\omega)]^{1/2}},$$

the phase response is

$$\forall \omega, \quad \angle H(\omega) = \tan^{-1} \frac{0.5 \sin(\omega)}{1 - 0.5 \cos(\omega)}.$$

The plots in figure 2 are for $0 \leq \omega \leq \pi$:

(f) In this case $x(n) \equiv e^{i0n}$, so $y(n) \equiv H(0)e^{i0n} = 2 = y_{ss}$.

7. Suppose x is a continuous-time periodic signal, with period p and exponential FS representation,

$$\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(ik\omega_0 t),$$

in which $\omega_0 = 2\pi/p$.

(a) Write down the formula for X_k in terms of x.

(b) Consider the signal y,

$$\forall t, \quad y(t) = x(\alpha t),$$

in which $\alpha > 0$ is some positive constant.

  i. Show that y is periodic and find its period q.

  ii. Suppose y has FS representation

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k \exp(k\omega_1 t),$$

  What is ω_1? Determine the Y_k in terms of the X_k.

Answer to 7 (a) The formula is

$$X_k = \frac{1}{p} \int_0^p x(t) e^{-ik\omega_0 t} dt.$$

(b) We want $y(t) = x(\alpha t) = y(t + q) = x(\alpha(t + q)) = x(t + p)$, so $\alpha q = p$ or $q = p/\alpha$. So the FS of y is

$$\begin{aligned} y(t) &= \sum_k Y_k e^{ik\omega_1 t} \\ &= \sum_k X_k e^{ik\alpha\omega_0 t} \end{aligned}$$

from which $\omega_1 = \alpha\omega_0$ and $Y_k = X_k$.

8. Give an example of a nonlinear, time-invariant system S that is **not** memoryless. Time is discrete.

(a) Show that S is nonlinear, time-invariant, and not memoryless.

(b) Suppose $x :$ *Ints* $\rightarrow$ *Reals* is periodic with period p. Let $y = S(x)$. Is y periodic?

(c) Suppose Q is another discrete-time, time-invariant system. Is the cascade composition $S \circ Q$ time-invariant? Give a proof or a counterexample.

(d) Define the system R by reversing time: $\forall x, n, R(x)(n) = S(x)(-n)$. Is R time-invariant? Why? If x is periodic as above and $w = R(x)$, is w periodic? Why.

Answer to 8 One possible system is

$$\forall x, \forall n, \quad S(x)(n) = [x(n-1)]^2.$$

(a) S is clearly nonlinear since, if $x(n-1) \neq 0$, $S(2x)(n) = 4[x(n-1)]^2 \neq 2[x(n-1)]^2$. S is time-invariant, since for any integer T,

$$S \circ D_T(x)(n) = [x(n-T-1)]^2 = D_T \circ S(x)(n).$$

S is not memoryless, because if it is there is f : *Reals* $\to$ *Reals* with

$$S(x)(n) = f(x(n)).$$

But this will not hold if we choose $x, n, n-1$ so that $x(n) = 0$ and $[x(n-1)]^2 \neq f(0)$.

(b) Yes it is periodic, since

$$S(x)(n+p) = D_{-p} \circ Sx(n) = S \circ D_{-p}(x)(n) = S(x)(n),$$

since $D_{-p}x = x$ because x is periodic with period p.

(c) The composiiton of any two time-invariant systems is periodic, since

$$D_T \circ (Q \circ S) = Q \circ D_T \circ S = (Q \circ S) \circ D_T.$$

(d) R is not time-invariant, because

$$\begin{aligned} D_T \circ R(x)(n) &= R(x)(n-T) = S(x)(-n+T) = [x(-n+T-1)]^2 \\ R \circ D_T(x)(n) &= S \circ D_T(x)(-n) = [D_T(x)(-n-1)]^2 = [x(-n-1-T)]^2. \end{aligned}$$

These two quantities are not equal for particular choices of x, n, T.

w is periodic with the same period p, because by part (b) $S(x)$ is periodic with period p, so

$$w(n+p) = S(x)(-n-p) = S(x)(-n) = R(x)(n) = w(n).$$

9. You are given three kinds of building blocks for discrete-time systems: one-unit delay; gains; and adders.

   (a) Use these building blocks to implement the system:

   $$y(n) = 0.5y(n-2) + x(n) + x(n-1). \tag{5}$$

   (b) Take the outputs of the delay elements as the state and give a $[A, b, c^T, d]$ representation of this system.

   (c) You are allowed to set the output of the delay elements to any value at time $n = 0$. Select these values so that the output of your implementation is the solution $y(n), n \geq 0$ for any input $x(n), n \geq 0$ and initial conditions: $y(-1) = 0.5, y(-2) = 0.8, x(-1) = 1$. Now suppose $x(0) = x(1) = x(2) = 0$. Calculate $y(0), y(1), y(2)$.

Answer to 9 (a) Figure 3 is one implementation.

(b) Taking $s(n) = [x(n-1)\ y(n-1)\ y(n-2)]^T$ and using (5) we get

$$\begin{aligned} s(n+1) &= \begin{bmatrix} x(n) \\ y(n) \\ y(n-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} s(n) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ y(n) &= [1\ 0\ 0.5]s(n) + 1 \times x(n) \end{aligned}$$

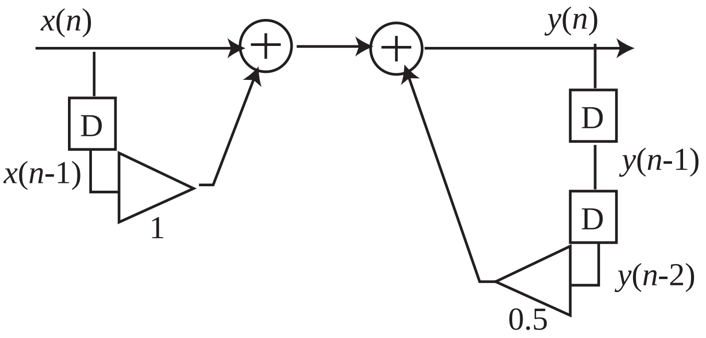


Figure 3: Implementation for problem 9

from which we can read off A, b, c, d.

(c) We take the initial state as $s(0) = [x(-1)\ y(-1)\ y(-2)]^T = [1\ 0.5\ 0.8]^T$. Then

$$\begin{aligned} y(0) &= c^T s(0) = [1\ 0\ 0.5]s(0) = 1.4 \\ y(1) &= c^T As(0) = 0.5^2 = 0.25 \\ y(2) &= c^T A^2 s(0) = 0.7 \end{aligned}$$

One can also get these directly from (5).

10. An integrator can be used as a building block: For any input x : *Reals*$_+$ $\to$ *Reals*, its output is:

$$\forall t \geq 0, \quad y(t) = y_0 + \int_0^t x(s)ds.$$

The 'initial condition' $y(0)$ can be set.

Use integrators, gains and adders to implement the system:

$$\frac{d^2y}{dt^2}(t) - y(t) = x(t), \tag{6}$$

with iniital condition $y(0) = 1, \dot{y}(0) = 0.4$.
Hint First convert a differential equation into an integral equation and then implement.

Answer to 10 Figure 4 shows the implementation

11. A periodic signal x : *Reals* $\to$ *Reals* is given by

$$\forall t, \quad x(t) = [1 + \cos(2\pi \times 10t)] \times \cos(2\pi \times 400t).$$

(a) What are the fundamental frequency ω_0 and period T_0 of x? Calculate the Fourier Series of x in the forms:

$$\begin{aligned} \forall t, \quad x(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) \\ &= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \end{aligned}$$

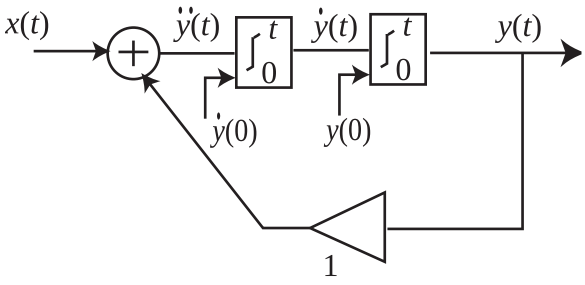


Figure 4: Implementation for problem 10

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Is $X_k = X^*_{-k}$?

(b) Suppose the LTI system S has frequency response

$$\forall \omega, \quad H(\omega) = \begin{cases} 1, & \text{if } 2\pi \times 395 \le |\omega| \le 2\pi \times 405 \\ 0, & \text{otherwise} \end{cases}$$

Plot the magnitude and phase response of H. Repeat part 11a for y.

Answer to 11 Using

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y),$$

gives

$$x(t) = \cos(2\pi \cdot 400t) + \frac{1}{2}\cos(2\pi \cdot 390t) + \frac{1}{2}\cos(2\pi \cdot 410t),$$

from which

(a) $\omega_0 = 2\pi \cdot 10$ rad/sec and $t_0 = 0.1$ sec. Also

$$A_{39} = 0.5, \quad A_{40} = 1.0, A_{41} = 0.5, \quad A_k = 0, \text{ else}; \forall k \phi_k = 0$$

and

$X_k = \frac{1}{2} A_{|k|} e^{\phi_k sgn(k)}$ in which $sgn(k) = 1, k \ge 0; = 0, k < 0$. So

$$X_{39} = X_{-39} = X_{41} = X_{-41} = 0.25; \quad X_{40} = X_{-40} = 0.5; \quad X_k = 0, \text{ else}.$$

(b) This system is a bandpass filter, in which only sinusoids with frequencies within specified range go through unchanged and the others become 0. Thus

$$\forall t, \quad y(t) = \cos(2\pi \cdot 400t); \quad \omega_0 = 2\pi \cdot 400 \text{ rad/sec}; \quad T_0 = \frac{1}{400} \text{ sec}.$$

So,

$$A_1 = 1; \quad A_k = 0, k \ne 1; \quad \phi_k = 0, \forall k,$$

$$X_1 = X_{-1} = 0.5; \quad X_k = 0 \text{ else}.$$

12. Give the ABCD state space representation of a discrete-time system with frequency response $H(\omega)$, where:

$$H(\omega) = \frac{2 + e^{-j\omega}}{1 - 3e^{-3j\omega}}$$

Hint: First find a difference equation which has the given frequency response. Then find the state space representation.

Answer to 12 From

$$H(\omega)[1 - 3e^{-3j\omega}] = 2 + e^{-j\omega}$$

we see that H is the frequency response of the difference equation

$$y(n) - 3y(n-3) = 2x(n) + x(n-1).$$

So we select

$$s(n) = \begin{bmatrix} x(n-1) \\ y(n-1) \\ y(n-2) \\ y(n-3) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix} \quad D = 2$$

13. You are given the signal $\forall t x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$ to use as input to a system with frequency response $H(\omega) = |\omega|$. Answer the following questions based on this setup.

    (a) Indicate the Fourier series expansion (in cosine format) of x by writing the nonzero values of A_0, A_k, and ϕ_k in the expansion $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$.

    (b) Indicate the Fourier series expansion (in complex exponential format) of $x(t)$ by writing the nonzero values of the complex coefficients X_k in the expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$.

    (c) Give y, the output of the system with input x.

Answer to 13 (a) First rewrite $x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$ in terms of cosines:

$$x(t) = 1 + \cos(20\pi t) + 2\cos(25\pi t + \frac{\pi}{2})$$

Next find the fundamental frequency. The largest frequency that evenly divides both 20π and 25π is $\omega_0 = 5\pi$. We rewrite $x(t)$ in terms of nonzero coefficients:

$$\begin{aligned} x(t) &= 1 + 1\cos(4(5\pi)t + 0) + 2\cos(5(5\pi)t + \frac{\pi}{2}) \\ &= A_0 + A_4\cos(4\omega_0 t + \phi_4) + A_5\cos(5\omega_0 t + \phi_5) \end{aligned}$$

We see from above that $A_0 = 1$, $A_4 = 1$, $\phi_4 = 0$, $A_5 = 2$, $\phi_5 = \frac{\pi}{2}$, and all other A_k and ϕ_k are zero.

(b) We can calculate the X_k's directly, but since we've already calculated the A_k's, let's use them to derive the X_k's. (See also page 302 in the text.) Note in particular that with complex exponentials, we have negative frequency and complex coefficients instead of phases, meaning that the X_k's are complex and k can be negative.

Recalling that

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2},$$

we can say that, for positive k:

$$\begin{aligned} A_k\cos(\omega_0 kt + \phi_k) &= \frac{A_k e^{j\phi_k}}{2} e^{j\omega_0 kt} + \frac{A_k e^{-j\phi_k}}{2} e^{-j\omega_0 kt} \\ &= X_k e^{j\omega_0 kt} + X_{-k} e^{j\omega_0 (-k)t} \end{aligned}$$

In our case, we have three nonzero A_k. We start with A_0. Since $\cos(0) = e^{j0} = 1$, we conclude that $X_0 = A_0$.

For A_4, we relate the frequency components at $\omega = \pm 4\omega_0$:

$$1\cos(4\omega_0 t) = \frac{1}{2}e^{4j\omega_0 t} + \frac{1}{2}e^{-4j\omega_0 t}$$

and conclude that $X_4 = 1/2$ and $X_{-4} = 1/2$.

And finally, for A_5 and ϕ_5, we relate the frequency components at $\omega = \pm 5\omega_0$.

$$\begin{aligned} 2\cos(5\omega_0 t) &= e^{j\pi/2}e^{5j\omega_0 t} + e^{-j\pi/2}e^{-5j\omega_0 t} \\ &= ie^{5j\omega_0 t} - ie^{-5j\omega_0 t} \end{aligned}$$

and conclude that $X_5 = i$ and $X_5 = -i$.

(c) We can either apply the frequency response to the eigenfunctions or we can look at $x(t)$

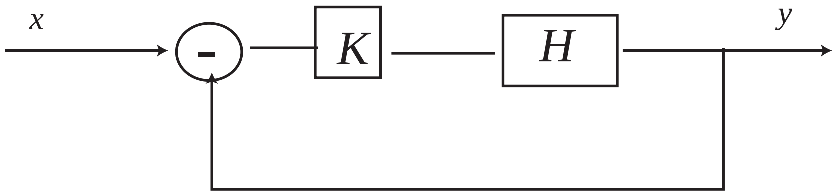

Figure 5: Feedback system for problem 14

component (i.e. the component at $\omega = 0$) gets completely attenuated (i.e. multiplied by 0). The other two components are scaled by the absolute value of their frequency, leading to:

$$\begin{aligned} y(t) &= (0)1 + (20\pi)\cos(20\pi t) - (25\pi)2\sin(25\pi t) \\ &= 20\pi\cos(20\pi t) - 50\pi\sin(25\pi t) \end{aligned}$$

If the frequency response had been more complicated, we may have preferred another approach:

We already have the complex exponential breakdown of the input signal, meaning that we know the input signal in terms of scaled eigenfunctions. We can therefore apply the frequency response:

$$\begin{aligned} y(t) &= H(0)X_0 \\ &\quad +X_4H(4\omega_0)e^{4j\omega_0 t} + X_{-4}H(-4\omega_0)e^{-4j\omega_0 t} \\ &\quad +X_5H(5\omega_0)e^{j5\omega_0 t} + X_{-5}H(-5\omega_0)e^{-5j\omega_0 t} \\ &= 0 + \frac{1}{2}|20\pi|e^{20\pi t} + \frac{1}{2}|-20\pi|e^{-20\pi t} + |25\pi|ie^{25\pi t} + |-25\pi|(-i)e^{-25\pi t} \\ &= 20\pi\frac{e^{20\pi t} + e^{-20\pi t}}{2} + 50\pi(i^2)\frac{e^{25\pi t} - e^{-25\pi t}}{2i} \\ &= 20\pi\frac{e^{20\pi t} + e^{-20\pi t}}{2} - 50\pi\frac{e^{25\pi t} - e^{-25\pi t}}{2i} \\ &= 20\pi\cos(20\pi t) - 50\pi\sin(25\pi t) \end{aligned}$$

which is the same result as with the other method.

14. In the negative feedback system of figure 5 assume that $H(\omega) = [1 + i\omega]^{-1}$. Let G be the closed-loop frequency response. For $K = 1, 10, 100$

    (a) Plot the magnitude and phase response of G; and

    (b) determine the bandwidth ω at which $\angle G(\omega) = \pi/4$.

Answer to 14 The closed loop frequency response is

$$\forall\omega, \quad G(\omega) = \frac{KH(\omega)}{1 + KH(\omega)} = \frac{K}{(K+1) + i\omega}.$$

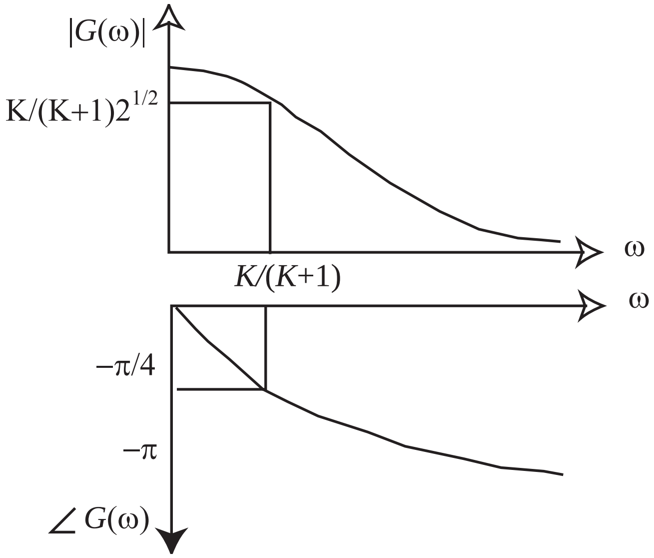

Figure 6: Frequency response for problem 14

(a) So

$$|G(\omega)| = \frac{K}{[(K+1)^2+\omega^2]^{1/2}}, \quad \angle G(\omega) = -\tan^{-1}\frac{\omega}{K+1}.$$

(b) See figure 6

15. Determine the 'gain' k and the guard so that the output of the hybrid system is as shown in figure 7

    Answer to 15 The gain and guard are given in figure 7.

16. Suppose we have a signal $x : Integers \to Reals$, which is zero for all negative time, that is,

    $$\forall k < 0, x(k) = 0.$$

    Suppose a signal $y : Integers \to Reals$ is obtained by filtering x as in Figure 8, with the following result:

    $$\begin{aligned} \forall k < 0, & \quad y(k) = 0 \\ \text{for } k = 0, & \quad y(k) = x(0) \\ \forall k > 0, & \quad y(k) = x(k-1) + x(k) \end{aligned}$$

    (a) Find the impulse response h of the system in Figure 8 and the frequency response H.

    (b) Suppose we receive the signal y, and we wish to recover the signal x. We can use a feedback connection to achieve this result. Design the impulse response g and frequency response G of the system used in feedback in Figure 9 so that the feedback system recovers the signal x.

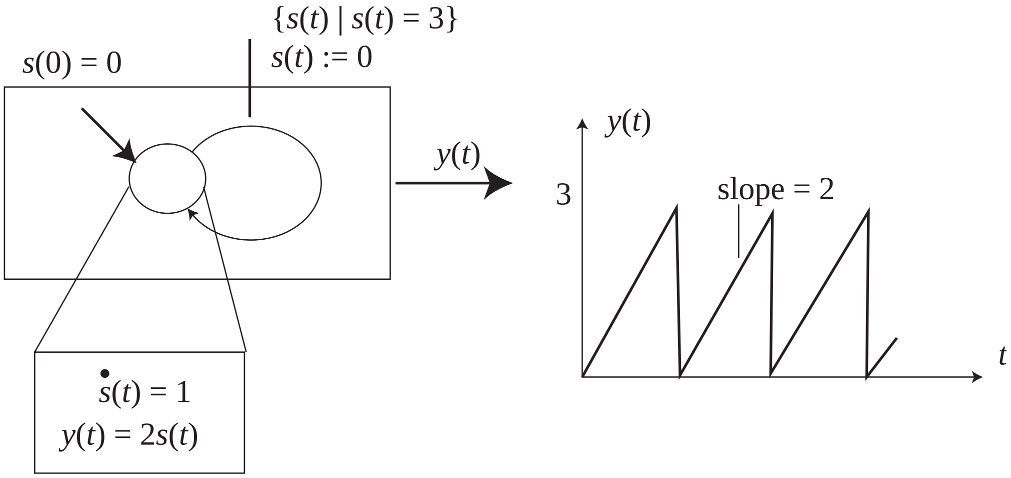

Figure 7: Hybrid system for problem 15

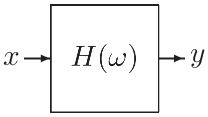

Figure 8: The filtering system.

(c) Find the impulse response f and the frequency response $F(\omega)$ of the overall feedback system from y to x in Figure 9.

Answer to 16

(a) Note that for all $k \in Integers$,

$$y(k) = x(k-1) + x(k).$$

Given the input δ, the output is then

$$y(k) = \delta(k-1) + \delta(k),$$

so the impulse response is

$$h(k) = \delta(k-1) + \delta(k).$$

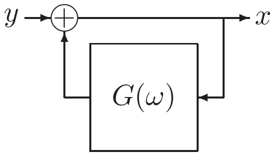

Figure 9: The feedback system.

Given the input $\forall k, x(k) = e^{j\omega k}$, the output is

$$y(k) = H(\omega)e^{j\omega k} = e^{j\omega(k-1)} + e^{j\omega k},$$

so,

$$H(\omega) = \frac{1}{e^{j\omega}} + 1.$$

(b) Since, for all $k \in Integers$,

$$y(k) = x(k-1) + x(k),$$

we know that

$$x(k) = y(k) - x(k-1).$$

Given input x, the output of G should then be $-D_1(x)$. Thus, given input δ, the output of G is $-D_1(\delta)$. For all $k \in Integers$,

$$g(k) = -\delta(k-1).$$

Given input $\forall k, x(k) = e^{j\omega k}$, the output is

$$G(\omega)e^{j\omega k} = e^{j\omega(k-1)},$$

so

$$G(\omega) = -\frac{1}{e^{j\omega}}.$$

(c) Note that given $y = \delta$, we get

$$\begin{aligned} \forall k < 0, &\quad x(k) = 0, \\ \forall k \geq 0, k \text{ even}, &\quad x(k) = 1, \\ \forall k \geq 0, k \text{ odd}, &\quad x(k) = -1, \end{aligned}$$

Thus, for all $k \in Integers$,

$$f(k) = \begin{cases} (-1)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Since we have a feedback system, we can use Equation (8.38) in the book so

$$F(\omega) = \frac{1}{1 - G(\omega)} = \frac{e^{j\omega}}{e^{j\omega} - 1}$$

Suppose that we have a SISO continuous time system of the following form:

$$\begin{aligned} \dot{s}(t) &= As(t) + bx(t), \\ y(t) &= c^T s(t). \end{aligned}$$

We decide to define a new state function $\tilde{s} : Reals \to Reals^N$, where

$$\forall t \in Reals, \tilde{s}(t) = Ts(t),$$

and T is an invertible $N \times N$ matrix. Find $\tilde{A}, \tilde{b}, \tilde{c}^T$ such that

$$\begin{aligned}\dot{\tilde{s}}(t) &= \tilde{A}\tilde{s}(t) + \tilde{b}x(t),\\ y(t) &= \tilde{c}^T\tilde{s}(t).\end{aligned}$$

In this case, we have transformed the state, but we still maintain the same input/output behavior.

Answer to 17 First note that

$$\dot{\tilde{s}}(t) = T\dot{s}(t).$$

Thus,

$$\dot{s}(t) = T^{-1}\dot{\tilde{s}}(t).$$

Then we get

$$\dot{\tilde{s}}(t) = TA\dot{s}(t) + Tbx(t) = TAT^{-1}\dot{\tilde{s}}(t) + Tbx(t).$$

Now

$$y(t) = c^T s(t) = c^T T^{-1}\tilde{s}(t).$$

Therefore

$$\begin{aligned}\tilde{A} &= TAT^{-1},\\ \tilde{b} &= Tb,\\ \tilde{c}^T &= c^T T^{-1}.\end{aligned}$$