EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 2
Department of Electrical Engineering and Computer Sciences 24 October 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Nut FIRST Name Nonlinear
Lab Time Midnight

- **(10 Points)** Print your name and lab time in legible, block lettering above (5 points) AND on the last page (5 points) where the grading table appears.

- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 12.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the twelve numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because *if we can't read it, we can't grade it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- We hope you do a *fantastic* job on this exam.

Basic Formulas:

Discrete Fourier Series (DFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \,,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\sum\limits_{k=\langle p \rangle}$ may denote $\sum\limits_{k=0}^{p-1}$ or $\sum\limits_{k=1}^{p}$.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT2.1 (20 Points) Consider a continuous-time system $\mathsf{F} : [\mathbb{R} \to \mathbb{C}] \to [\mathbb{R} \to \mathbb{C}]$ having input signal x and output signal y, as shown below:

$$x \longrightarrow \boxed{\mathsf{F}} \longrightarrow y$$

This system takes the real part of its input signal:

$$y = F(x) = \mathsf{Re}(x).$$

In other words,

$$\forall t \in \mathbb{R}, \quad y(t) = \mathsf{Re}(x(t)),$$

where $\mathsf{Re}(\cdot)$ denotes taking the real part of a number. For each part below, you must explain your reasoning succinctly, but clearly and convincingly.

(a) Select the strongest true assertion from the list below.

- $\boxed{\text{(i) The system must be memoryless.}}$
- (ii) The system could be memoryless, but does not have to be.
- (iii) The system cannot be memoryless.

$y(t)$ depends only on $x(t)$. A function $f: \mathbb{C} \to \mathbb{C}$ exists such that $y(t) = f(x(t))$. That function f is, in fact, $f: \mathbb{C} \to \mathbb{C}$

$$\forall \alpha \in \mathbb{C}, \quad f(\alpha) = Re(\alpha)$$

(b) Select the strongest true assertion from the list below.

- $\boxed{\text{(i) The system must be causal.}}$
- (ii) The system could be causal, but does not have to be.
- (iii) The system cannot be causal.

- A memoryless system must also be causal
- Alternatively, you can say if $x_1(\tau) = x_2(\tau), \ \forall \tau \le t$ then $y_1(\tau) = Re(x_1(\tau)) = Re(x_2(\tau)) = y_2(\tau), \ \forall \tau \le t$.

So, the system must be causal.

(c) Select the strongest true assertion from the list below.

$\boxed{\text{(i) The system must be time invariant.}}$

(ii) The system could be time invariant, but does not have to be.

(iii) The system cannot be time invariant.

- We know (x, y) is a behavior of F.
  Let $\hat{x}$ be defined such that $\hat{x}(t) = x(t - t_0)$ $\forall t$, $\exists t_0 \in \mathbb{R}$ (arbitrary)
  The corresponding response $\hat{y}$ is characterized by
  $\hat{y}(t) = Re(\hat{x}(t)) = Re(x(t-t_0)) = y(t-t_0)$. $\Longrightarrow$
  F is time invariant.
- Alternatively, every memoryless system, according to our definition, is time invariant.

(d) Select the strongest true assertion from the list below.

(i) The system must be linear.

(ii) The system could be linear, but does not have to be.

$\boxed{\text{(iii) The system cannot be linear.}}$

The system is additive, but does not satisfy homogeneity.

- If (x_1, y_1) and (x_2, y_2) are behaviors of the system, then
  $y_1(t) = Re(x_1(t))$ and $y_2(t) = Re(x_2(t))$. Clearly,
  $y_1(t) + y_2(t) = Re(x_1(t)) + Re(x_2(t)) = Re(x_1(t) + x_2(t))$, which
  means $(x_1 + x_2, y_1 + y_2)$ is a behavior.
- To discuss homogeneity, let (x, y) be a behavior of F.
  Let $\hat{x} = \alpha x$, $\exists \alpha \in \mathbb{C}$. Then $\hat{y}(t) = Re(\alpha x(t))$. The
  right-hand side is <u>not</u> equal to $\alpha Re(x(t)) \triangleq \alpha y(t)$
  unless $\alpha \in \mathbb{R}$. Any complex α ruins the party.

MT2.2 (25 Points) The unit-step response[1] s of a discrete-time linear, time-invariant system is given by:

$$\forall n \in \mathbb{Z}, \quad s(n) = (n+1)\,u(n)\,,$$

where u is the unit-step signal characterized as follows:

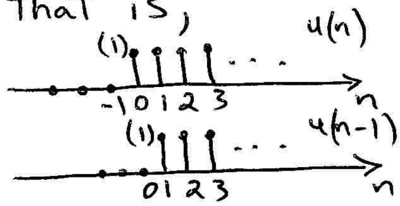
$$\forall n \in \mathbb{Z}, \quad u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0\,. \end{cases}$$

Explain your reasoning for each part succinctly, but clearly and convincingly.

(a) Determine and provide a well-labeled sketch of h, the impulse response of the system.

The DT impulse can be writen as the difference between shifted unit step functions; that is,

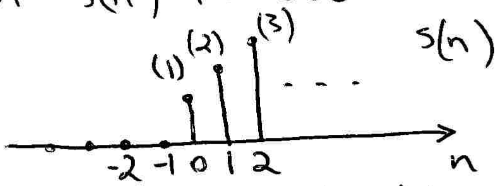
$\delta(n) = u(n) - u(n-1)$



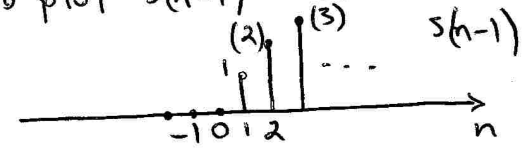
Since the system is LTI,

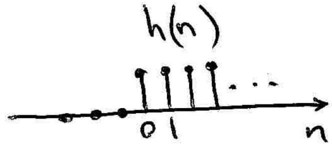
$h(n) = s(n) - s(n-1)$

Plot s(n) to see how things work:



Also plot s(n-1) and subtract to get →





$h(n) = u(n)$

[1]Recall that the unit-step response of a system is, as the name suggests, the response of the system to the unit-step input signal.

(b) Select the strongest true assertion from the list below.

(i) The system must be memoryless.

(ii) The system could be memoryless, but does not have to be.

(iii) The system cannot be memoryless.

No! Note that the unit-step input has the property that $u(0)=u(1)=u(2)=\cdots=1$

However, the output s is such that

$$s(0) \neq s(1) \neq s(2) \neq \cdots$$

Hence, the system can't be memoryless.

Further exploration: Convince yourself that an LTI system is memoryless if, and only if, its impulse response $h(n)=\alpha\,\delta(n)$, $\exists \alpha \in \mathbb{C}$.

(c) Determine a simple expression for

$$\sum_{m=-\infty}^{n} h(m).$$

Hint: Your answer will depend on n. You should be able to solve this part even without knowing the impulse response h from part (a).

The step response "s" is obtained by convolving the unit step function "u" with the impulse response "h." That is: $s = h * u$. Therefore,

$$s(n)=(h*u)(n)=\sum_{m=-\infty}^{\infty} h(m)\,u(n-m)$$

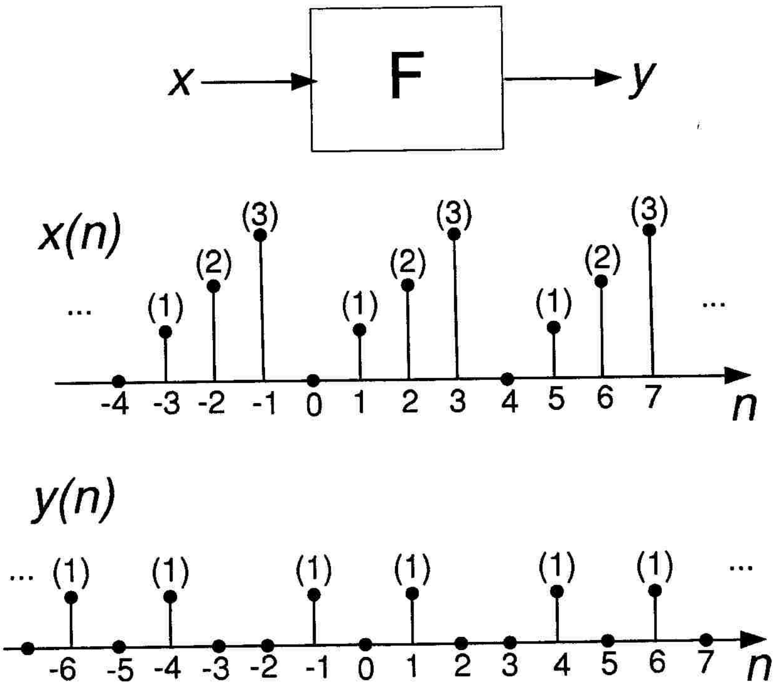
but $u(n-m)=0$ for $m>n$

$=1$ for $m \leq n$

$$s(n)=\sum_{m=-\infty}^{n} h(m) \Longrightarrow \boxed{\sum_{m=-\infty}^{n} h(m) = s(n) = (n+1)\,u(n)}$$

Moral of the story: Convolving a function/signal with the unit step produces a cumulative sum of the function.

MT2.3 (25 Points) Consider a discrete-time system $\mathsf{F} : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ having a periodic input signal x and a corresponding periodic output signal y, as shown below.



(a) Determine (p_x, ω_x) and (p_y, ω_y), the period and fundamental frequency of x and y, respectively.

$p_x = 4 \implies \omega_x = \frac{2\pi}{4} = \frac{\pi}{2}$

$p_y = 5 \implies \omega_y = \frac{2\pi}{5}$

(b) Determine the complex exponential discrete Fourier series (DFS) representation of the output signal y. In particular, determine a simple expression for the coefficients Y_k in the DFS expansion

$$Y_k = \frac{1}{p_y} \sum_{n=\langle p_y \rangle} y(n)\, e^{-ik\omega_y n}$$

$$Y_k = \frac{1}{5} \sum_{n=-2}^{2} y(n)\, e^{-ik\frac{2\pi}{5}n}$$

Only two terms, $y(1)$ and $y(-1)$, are nonzero. We know $y(1) = y(-1) = 1 \Rightarrow Y_k = \frac{1}{5}\left[e^{ik\frac{2\pi}{5}} + e^{-ik\frac{2\pi}{5}}\right] = \frac{2}{5}\cos\left(\frac{2\pi}{5}k\right)$

$$Y_k = \frac{2}{5}\cos\left(\frac{2\pi}{5}k\right)$$

$k = \langle 5 \rangle$
(e.g., $-2, -1, 0, 1, 2$)
or $0, 1, 2, 3, 4$
etc.

(c) Select the strongest true assertion from the list below. Explain your reasoning succinctly, but clearly and convincingly.

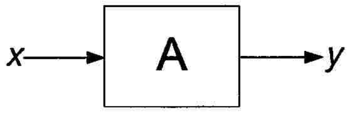
(i) The system must be LTI.

(ii) The system could be LTI, but does not have to be.

(iii) The system cannot be LTI.

The output contains frequencies that are not present in the input. For example, $\omega_y = \frac{2\pi}{5}$ is nonexistent in the input signal x.
We know LTI systems can only modify (e.g., amplify or attenuate) frequency components present in their inputs; they cannot create new frequencies.

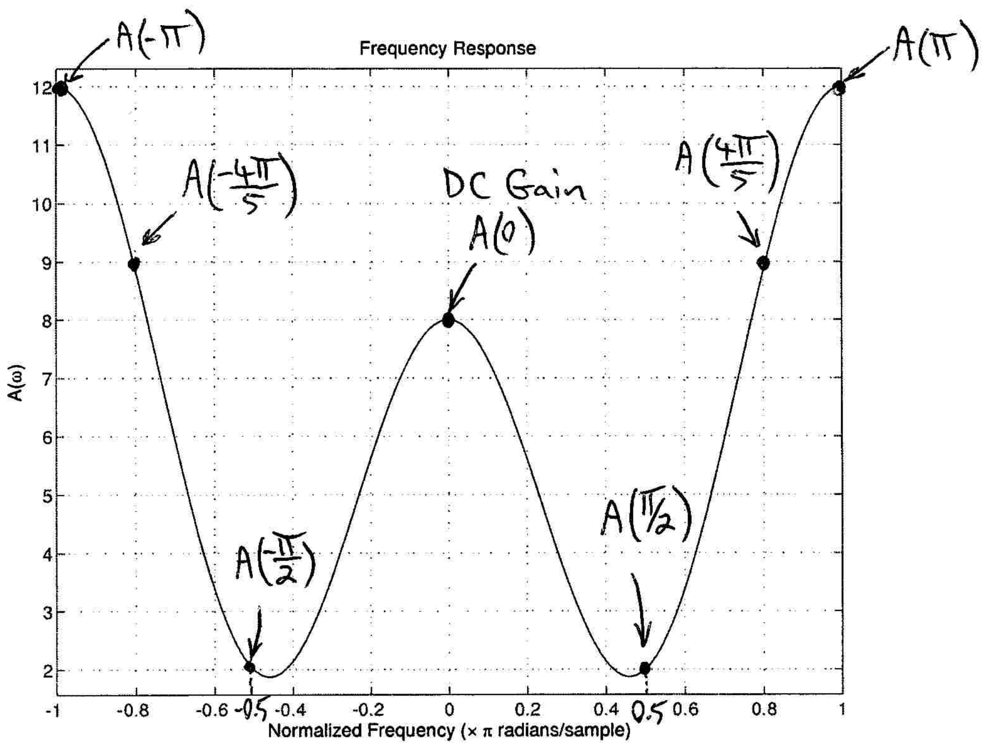
MT2.4 (20 Points) Consider a discrete-time LTI filter $\mathsf{A} : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ having impulse response a and frequency response A. The figure below is a graphical, input-output depiction of the filter:



Recall that the frequency response and impulse response are related as follows:

$$\forall \omega \in \mathbb{R}, \quad A(\omega) = \sum_{n=-\infty}^{\infty} a(n)\, e^{-i\omega n}.$$

The figure below depicts $A(\omega), \forall \omega \in [-\pi, +\pi]$. Notice that for this particular filter, $A(\omega)$ is real-valued at all frequencies.



The frequency axis in the figure is normalized by π; hence, for example, the normalized frequencies 0.5 and 1 refer to $\omega = \pi/2$ and $\omega = \pi$ radians per sample, respectively.

Determine a reasonable and simple (possibly approximate) expression for the output y of the filter, if the input x is:

$$\forall n \in \mathbb{Z}, \quad x(n) = e^{i\pi/3} + \cos\left(\frac{4\pi}{5}n\right) + (-1)^n + i^n .$$

Note that there is no "n" in the first term. This is not a typographical error.

- $e^{i\frac{\pi}{3}} = e^{i\frac{\pi}{3}} \cdot 1 \Longrightarrow$ Look @ DC gain $A(0)$ to see what happens to this zero frequency component: $A(0)e^{i\pi/3} = 8e^{i\pi/3}$
- $\cos\left(\frac{4\pi}{5}n\right) = \frac{1}{2}\left(e^{i\frac{4\pi}{5}n} + e^{-i\frac{4\pi}{5}n}\right) \longrightarrow$ Look @ $A\left(\frac{4\pi}{5}\right)$ and $A\left(-\frac{4\pi}{5}\right)$ to determine the corresponding output: $9 = A\left(\frac{4\pi}{5}\right) = A(0.8\pi) = A(-0.8\pi) = A\left(\frac{-4\pi}{5}\right)$
Therefore, the contribution of $\cos\left(\frac{4\pi}{5}n\right)$ to the response is $9\cos\left(\frac{4\pi}{5}n\right)$.
- $(-1)^n = e^{i\pi n} \Longrightarrow$ Output contribution is $A(\pi)(-1)^n = 12(-1)^n$
- $i^n = e^{i\frac{\pi}{2}n} \Longrightarrow$ Output contribution is $A\left(\frac{\pi}{2}\right)i^n = 2i^n$

Total response is obtained by superposition: $\boxed{y(n) = 8e^{i\frac{\pi}{3}} + 9\cos\left(\frac{4\pi}{5}n\right) + 12(-1)^n + 2i^n}$

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

MT2.5 (15 Points) The impulse response h of a discrete-time LTI system is given by:

$$\forall n \in \mathbb{Z}, \quad h(n) = \left(\frac{1}{2}\right)^n u(n),$$

where u is the unit-step function.

Main lesson of this problem:
An LTI system is causal $\iff h(n)=0, \forall n<0$

(a) Select the strongest true assertion from the list below.

$\boxed{\text{(i) The system must be causal.}}$

(ii) The system could be causal, but does not have to be.

(iii) The system cannot be causal.

$h(m)=0 \quad m<0$

$$y(n) = (h * x)(n) = \sum_{m=-\infty}^{\infty} h(m)\,x(n-m) = \sum_{m=0}^{\infty} h(m)\,x(n-m)$$

Note that we can write the output as $y(n) = h(0)x(n) + h(1)\,x(n-1) + \cdots$

That is, no "future" value of the input enters the expression, e.g., there are no $x(n+1), x(n+2), \ldots$ terms $\Longrightarrow$ System is causal.

(b) Determine a simple expression for the frequency response H of the system. Recall that the frequency response and impulse response are related as follows:

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}.$$

Hint: You may find the following helpful. If $|\alpha| < 1$, then $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-i\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-i\omega}\right)^n$$

$$\Rightarrow H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-i\omega}}$$

1

LAST Name Nut FIRST Name Nonlinear

Lab Time Midnight

Problem	Points	Your Score
Name	10	10
1	20	20
2	25	25
3	25	25
4	20	20
5	15	15
Total	**115**	115