EECS 20. Midterm 1. 2 October 10, 1998. Sample Problems

Note. Each problem is annotated with the letter **E, T, C** which stands for exercise, requires some thought, requires some conceptualization. Problems labeled **E** are usually mechanical, those labeled **T** require a plan of attack, those labeled **C** usually have more than one defensible answer.

Complex exponentials

1. **E** Define $x : Reals \to Reals$

$$\forall t \in Reals, x(t) = 5\cos(\omega_0 t + 1/2\pi) + 5\cos(\omega_0 t - 1/6\pi) + 5\cos(\omega_0 t - 2/3\pi).$$

Find A and ϕ so that

$$\forall t \in Reals, x(t) = A\cos(\omega_0 t + \phi).$$

2. **E** Find θ so that

$$Re[(1+i)\exp i\theta] = -1.$$

3. **E** Define $x : Reals \to Reals$

$$\forall t \in Reals, x(t) = \sin(\omega_0 t + 1/4\pi).$$

Find $A \in Comps$ so that

$$\forall t \in Reals, x(t) = A\exp(i\omega_0 t) + A^* \exp(-i\omega_0 t),$$

where A^* is the complex conjugate of A.

4. **T** Suppose $A_k \in Comps$, $\omega_k \in Reals$, $k = 1, 2$, such that

$$\forall t \in Reals, A_1 \exp(i\omega_1 t) = A_2 \exp(i\omega_2 t). \tag{1}$$

Prove that $A_1 = A_2$ and $\omega_1 = \omega_2$.
Hint. Evaluate the two sides of (1) at $t = 0$ and evaluate their derivates at $t = 0$.

This result shows that in order for two complex exponential signals to be equal, their frequencies, phases, and amplitudes must be equal. More interestingly, this result can be used to show that if a signal can be described as a sum of complex exponential signals, then that description is unique. There is no other sum of complex exponentials (one involving different frequencies, phases, or amplitudes) that will also describe the signal. In particular, the Fourier series representation of a periodic signal is unique, stated below as a Theorem.

Theorem Suppose $x : Reals \to Reals$ is a periodic function with period p such that $\forall t \in Reals$,

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k) \\ &= B_0 + \sum_{k=1}^{\infty} B_k \cos(k\omega_0 t + \psi_k) \end{aligned}$$

where $\omega_0 = 2\pi/p$. Then

$$\forall k \geq 0, A_k = B_k, \text{ and } \phi_k = \psi_k$$

It is this uniqueness result that allows us to refer to *the* frequency content of a signal.

5. **E** Plot the function $s : Reals \to Reals$

$$\forall x \in Reals, s(x) = Im[\exp(-x + i2\pi x)]$$

6. **T** When a transmitter sends a radio signal to a reciever, the received signal consists of the direct path plus several reflected paths. In Figure 1, the transmitter is on a tower at the right of the figure, the receiver is on the vehicle, and there are three paths: the direct path is l_0 meters long, the path reflected from a hill (the little triangle in the middle) is l_1 meters long, and the path reflected from the building (the rectangle at the left) is l_2 meters long.

   Suppose the transmitted signal is a f Hz sinusoid $x : Reals \to Reals$,

$$\forall t \in Reals, x(t) = A\cos(2\pi f t)$$

   So the received signal is y. $\forall t \in Reals$,

$$y(t) = \alpha_0 A\cos(2\pi f(t - \frac{l_0}{c})) + \alpha_1 A\cos(2\pi f(t - \frac{l_1}{c})) + \alpha_2 A\cos(2\pi f(t - \frac{l_2}{c})) \qquad (2)$$

   Here, $0 \leq \alpha_i \leq 1$ are numbers that represent the attenuation (or reduction in signal power) of the signal and $c = 3 \times 10^8$ m/s is the speed of light.[1] Answer the following questions.

   (a) Explain why the description of y given in (2) is a reasonable model of the received signal.

   (b) What would be the description if instead of the 3 paths as shown in Figure 1, there were 10 paths (one direct and 9 reflected).

   (c) The signals received over the different paths cause different phase shifts, ϕ_i, so the signal y can also be written as

$$\forall t \in Reals, y(t) = \sum_{i=0}^{2} \alpha_i A\cos(2\pi f(t) - \phi_i)$$

   What are the ϕ_i?

   (d) Let $\Phi = \max\{\phi_1 - \phi_0, \phi_2 - \phi_0\}$ be the largest difference in the phase of the received signals and let $L = \max\{l_1 - l_0, l_2 - l_0\}$ be the maximum path length difference. What is the relation between Φ, L, f?

   (e) Suppose for simplicity that there is only one reflected path of distance l_1, i.e. take $\alpha_2 = 0$ in the expressions above. Then $\Phi = \phi_1 - \phi_0$. When $\Phi = \pi$, the reflected signal is said to *destroy* the direct signal. Explain why the term "destroy" is appropriate.

[1] In reality, the reflections are more complicated than the model here.

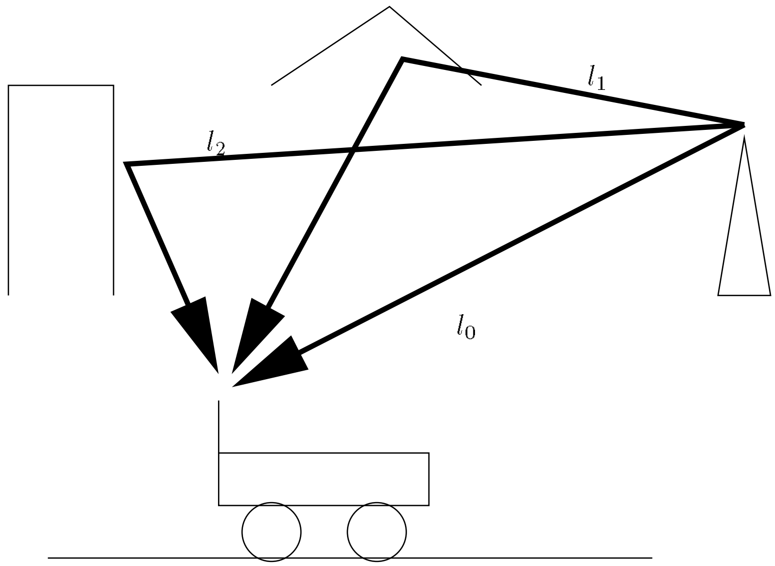

Figure 1: The direct and two reflected paths from transmitter to receiver

(f) In the context of mobile radio shown in the figure, $L \leq 500$m. For what values of f is $\Phi \leq \pi/10$? (Note that if $\Phi \leq \pi/10$ the signals will not act destructively.)

(g) Derive an expression that relates the frequencies f that are destroyed to the path length difference $L = l_1 - l_0$.

Sets

1. **E** Draw the product set $\{M, Tu, W, Th, F\} \times [8.00, 18.00]$ and indicate on that set when your lab section meets.

2. **E** Draw the following sets

   (a) $\{(x, y) \in Reals^2 \mid y = 2x\}$.

   (b) $\{(x, y) \in Reals^2 \mid xy = 0\}$.

   (c) $\{(x, y) \in Reals^2 \mid y - x^2 = 0\}$.

   (d) $\{z \in Comps \mid |z| = 1\}$, where $|z|$ is the magnitude of z.

   (e) **T** $\{z \in Comps \mid z^6 = 1 + 0i\}$.

   (f) $\{z \in Comps \mid z^2 + 2 = 0\}$

3. **E** Suppose licence plates are of the form $XXXYYY$ where the X can be any uppercase letter in the alphabet and Y can be any digit. How many distinct licence plates can there be?
   Hint. Recall a homework exercise to figure out the number of elements of a product set $X_1 \times \cdots \times X_n$.

Predicates

1. **E** Consider the predicate S,
$$S = [P \wedge Q] \vee \neg R$$
Find the truth value of S for the following values of P, Q, R.

P	Q	R	S
True	False	False	
False	True	False	
True	False	True	

2. **T** The set $X = \{x \in Reals \mid (x-1)(x-2) \geq 0\}$ can also be written as a union of intervals $X = (-\infty, 1] \cup [2, \infty)$. Express Y as a union of intervals, where
$$Y = \{x \in Reals \mid (x-1)(x-2)(x-3) \geq 0\}$$

3. **E** Sketch the set $\{(x, y) \in Reals^2 \mid (x \geq y) \wedge (x \leq 1) \wedge (y \geq 0)\}$

4. **T** The following sequence of statements is a complete context.
Let
$$x = 5, y = 6 \tag{3}$$
Then,
$$x \geq y \tag{4}$$
Now let
$$Z = \{z \in Reals \mid z \geq x + y\} \tag{5}$$
Then
$$x \in Z \tag{6}$$
Let
$$w = \text{ smallest non-negative number in } Z \tag{7}$$
Answer the following:
   (a) Are the two expressions in (3) both assignments or assertions?
   (b) Is the expression (4) an assertion or a predicate?
   (c) Is the equality in (5) an assignment or an assertion?
   (d) Is the expression "$z \geq x + y$" in (5) an assertion or a predicate?
   (e) Is (6) an assertion or a predicate?
   (f) Is (7) an assignment or an assertion?

Functions

1. **T** The function informally described as one that takes 2 numbers and returns their average, is mathematically defined as $f : Reals^2 \to Reals$,

$$f(x_1, x_2) = 1/2(x_1 + x_2)$$

Express in similar mathematical form functions that are informally described below.

   (a) It takes 10 numbers and returns their sum.

   (b) It takes any finite set and returns the number of elements of that set.

   (c) It takes 10 distinct numbers and returns those numbers in a list in decreasing order.

2. **T** How would you represent as a table the function $Score$ which assigns 4 numbers to each student in class: his or her grades in the homeworks, 2 midterms, and 1 final. Suppose there are 50 students in the class. Consider the function $average : Reals^{50} \to Reals$,

$$\forall x = (x_1, \cdots, x_{50}) \in Reals^{50}, average(x) = \frac{1}{50}\sum_{i=0}^{50} x_i$$

Write the average grades in the homework, mideterms and final using $Score$, $average$, and function composition.

3. **T** $OddParity : Bin^4 \to Bin$ is a function such that $OddParity(b_1, \cdots, b_4) = 1$ if and only if an odd number of the b_i equal 1. Build $OddParity$ out of AND, OR, and NOT gates.

4. **T** Consider the function $AmIPrime : Nats \to \{Yes, No\}, \forall n \in Nats, AmIPrime(n) = 1$ or 0, accordingly as n is or is not a prime number. This is a declarative definition of $AmIPrime$ since we don't know how to evaluate whether a number is a prime. What would be an imperative definition of $AmIPrime$?

Definition of signal space A signal is a function. We have studied signals whose domain is time, or space, or a set of indices, and whose range are physical attributes like pressure, voltage, speed, light intensithy, or more abstract entities like symbols or binary values. Mathematically, we model a signal space as a collection of functions with a common range and a common domain. For example, $Sounds = [Time \to Pressure]$. Propose mathematical models for the signal space corresponding to the following intuitive descriptions.

1. **C** The button presses that you input into your microwave oven.

2. **C** The response of your microwave oven.

3. **C** The actions you take when withdrawing cash from an ATM machine.

4. **C** The steps you follow when you are cooking something.

5. **C** The inputs you apply when riding your bicycle.

6. **C** The motion of your bicycle.

7. **C** The signal that your radio attenna receives.

8. **C** The signal generated by the amplifier in your tuner.

9. **C** The signal input to your speaker.

10. **C** The output of your speaker.

11. **C** The keyboard strokes that you make at your PC terminal.

12. **C** The actions of the salesperson at the checkout counter at your local grocery store.

13. **C** The buttons you press with your TV remote control.

Definition of systems

An equalizer is represented by a function $Equalizer : Sounds \to Sounds$ where $Sounds = [Time \to Pressure]$. Represent the following systems as functions or mappings from the space of input signals to the space of output signals. In each case give a name for the function and specify the space of input signals and the space of output signals. These signal spaces should be in the standard form $[X \to Y]$, and you must identify X and Y.

1. **C** A voice recognition system which converts voice into text.

2. **C** An optical character recognition (OCR) system which converts an image of a page into text.

3. **C** A rocket whose input is thrust and output is vertical position.

4. **C** A steel plant with two inputs, coal and iron ore, and output is steel.

5. **C** A paper plant with two inputs, wood pulp and water, and output is paper.

6. **C** An engine whose input is gasoline, and output is torque and carbon dioxide.

Frequency decomposition

1. **E** Which of the following signals is periodic with a period greater than zero, and what is that period? All functions are denoted $x : Reals \to Comps$. The time is measured in seconds, and so the period is in seconds.

   (a) $\forall t \in Reals, x(t) = 10\sin(2\pi t) + (10 + 2i)\cos(2\pi t)$
   (b) $\forall t \in Reals, x(t) = \sin(2\pi t) + \sin(\sqrt{2}\pi t)$
   (c) $\forall t \in Reals, x(t) = \sin(2\sqrt{2}\pi t) + \sin(\sqrt{2}\pi t)$

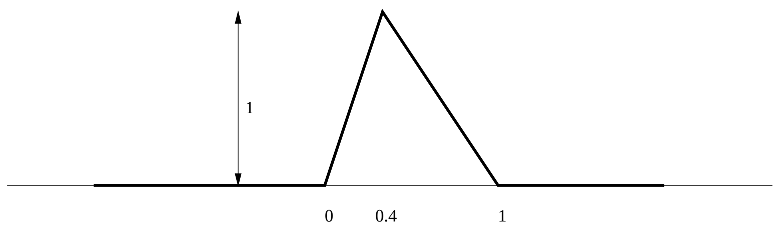

Figure 2: The graph of x

2. **T** The function $x : Reals \to Reals$ is given by its graph shown below Note that $\forall t \notin [0,1], x(t) = 0$, and $x(0.4) = 1$. Define y by

$$\forall t \in Reals, y(t) = \sum_{k=-\infty}^{\infty} x(t - kp)$$

where p is the period.

   (a) Prove that y is periodic with period p, i.e.

$$\forall t \in Reals, y(t) = y(t + p).$$

   (b) Plot y for $p = 1$.
   (c) Plot y for $p = 2$.
   (d) Plot y for $p = 0.8$.
   (e) Plot y for $p = 0.5$.
   (f) Suppose the function z is obtained by advancing x by 0.4, i.e.

$$\forall t, \ z(t) = x(t + 0.4).$$

   Define w by

$$\forall t \in Reals, w(t) = \sum_{k=-\infty}^{\infty} z(t - kp)$$

   What is the relation between w and y. Use this relation to plot w for $p = 1$.

3. **T** Suppose $x : Reals \to Reals$ is a periodic signal with period p, i.e.

$$\forall t \in Reals, x(t) = x(t + p).$$

Let $f : Reals \to Reals$ be any function, and define the signal $y : Reals \to Reals$ by $y = f \circ x$, i.e.

$$\forall t \in Reals, y(t) = f(y(t)).$$

   (a) Prove that y is periodic with period p.

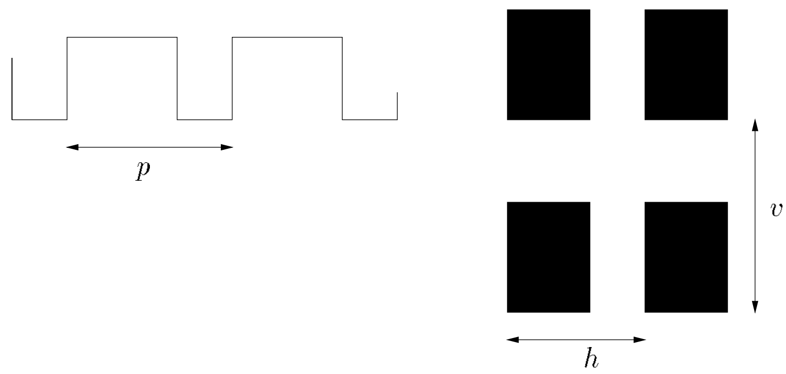

Figure 3: A periodic square wave (left) and a periodic pattern (right)

(b) Suppose $\forall t \in Reals, x(t) = \sin(2\pi t)$. Suppose f is the sign function, $\forall x \in Reals$,

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Plot y.

(c) Suppose $\forall t \in Reals, x(t) = \sin(2\pi t)$. Suppose f is the square function,$\forall x \in Reals$, $f(x) = x^2$. Plot y.

4. **C** Suppose the periodic square wave shown on the left in Figure 3 has the Fourier series representation

$$A_0 + \sum_{k=0}^{\infty} A_k \cos(2\pi kt/p + \phi_k)$$

Use this to obtain a Fourier series representation of the two-dimensional pattern of rectangles on the right. Note that the vertical and horizontal periods l, w are different.