EECS 20. Solutions to Practice Problems No. 2.

1. Note that

$$1 + i = \sqrt{2}e^{i\pi/4}.$$

So we can write

$$\begin{aligned} Re\{(1+i)e^{i\theta}\} &= \sqrt{2}Re\{e^{i\pi/4}e^{i\theta}\} \\ &= \sqrt{2}Re\{e^{i(\theta+\pi/4)}\} \\ &= \sqrt{2}\cos(\theta+\pi/4). \end{aligned}$$

So our task is to find θ so that

$$\sqrt{2}\cos(\theta+\pi/4) = -1.$$

So

$$\theta = \cos^{-1}(-1/\sqrt{2}) - \pi/4.$$

There are two values for $\cos^{-1}(-1/\sqrt{2})$ in the range 0 to 2π, namely $3\pi/4$ and $5\pi/4$, which yields the two possible solutions

$$\theta = \pi/2 \quad \text{and} \quad \pi.$$

We can check both of these by substituting back into the original equation.

2. Find 5 roots of $z^5 + 2$ by solving

$$z^5 + 2 = 0$$

or

$$z^5 = -2.$$

Let $z = re^{i\theta}$ be the polar form representation, and solve

$$r^5 e^{i5\theta} = -2 = 2e^{i\pi}.$$

Since the magnitudes on both sides must be equal, and r is a nonnegative real number,

$$r = 2^{(1/5)}.$$

Moreover, the arguments must be equal, modulo 2π,

$$5\theta \text{ mod } 2\pi = \pi$$

so

$$\theta = \pi/5,\ 3\pi/5,\ \pi,\ 7\pi/5,\ 9\pi/5,\ \text{ or }\ 11\pi/5.$$

So the solutions are $re^{i\theta}$ with the specified value of r and values of θ.

3. Let $x = \sqrt{1+i}$ mean any x such that $x^2 = 1 + i = \sqrt{2}e^{i\pi/4}$. So,

$$x = 2^{(1/4)}e^{i\pi/8} \quad \text{or} \quad 2^{(1/4)}e^{i(\pi+\pi/8)}.$$

Just as with the square root of real numbers, there are two answers. Alternatively, it is defensible to define $\sqrt{1+i}$ to be only the first of these, since the second is the negative of the first.

More generally, define $x = \sqrt{z}$ to be any x satisfying $x^2 = z$. Write $z = re^{i\theta}$ in polar form, so

$$x = \sqrt{r}e^{i\theta/2} \quad \text{or} \sqrt{r}e^{i(\pi=\theta/2)}.$$

Again, we could take the first of these to be the definition, since the second is the negative of the first.

4. By the given definition, $\log(1)$ is all w satisfying

$$e^w = 1$$

so

$$\log(1) = 0,\ i2\pi,\ i4\pi,\ \cdots.$$

Similarly, $\log(-1)$ is all w satisfying

$$e^w = -1$$

so

$$\log(-1) = i\pi,\ i3\pi,\ i5\pi,\ \cdots.$$

Similar methods yield

$$\log(i) = i\pi/2,\ i5\pi/2,\ i9\pi/2,\ \cdots.$$

and

$$\log(-i) = i7\pi/2,\ i5\pi/2,\ i11\pi/2,\ \cdots.$$

Finally, $\log(1+i)$ is all w satisfying

$$e^w = 1 + i = \sqrt{2}e^{i\pi/4}.$$

Write this as

$$e^{\mathit{Re}\{w\}}e^{i\mathit{Im}\{w\}} = \sqrt{2}e^{i\pi/4},$$

so

$$Re\{w\} = \log(\sqrt{2}),$$

where this is the ordinary natural logarithm, and

$$Im\{w\} = \pi/4 + 2\pi n$$

for all $n \in$ *Ints*. More generally, if $z = re^{i\theta}$, then

$$Re\{\log(z)\} = \log(r),$$

and

$$Im\{\log(z)\} = \theta + 2\pi n$$

for all $n \in$ *Ints*.

5. (a) Write

$$\begin{aligned} \cos^3(\omega t) &= 1/8[e^{i\omega t} + e^{-i\omega t}]^3 \\ &= 1/8[e^{3i\omega t} + e^{-3i\omega t} + 3e^{i\omega t} + 3e^{-\omega t}] \\ &= 1/4[\cos(3\omega t) + 3\cos(\omega t).] \end{aligned}$$

(b) The distinct roots are:

$$\{2^{1/5} \times e^{i2\pi k/5} \mid k = 0, 1, 2, 3, 4\}.$$

(c) Since

$$\sin(\omega t + \theta) = \sin(\omega t)\cos(\theta) + \cos(\omega t)\sin(\theta) = \sin(\omega t) + \cos(\omega t),$$

we must have $A\cos(\theta) = A\sin(\theta) = 1$ which gives $A = \sqrt{2}, \theta = \pi/4$.

(d) We have

$$\frac{1 + j\omega}{1 - j\omega} = \frac{(1 + j\omega)^2}{1 + \omega^2} = \frac{1 - \omega^2}{1 + \omega^2} + j\frac{2\omega}{1 + \omega^2},$$

and

$$\frac{1 + j\omega}{1 - j\omega} = 1 \times e^{j2\tan^{-1}\omega}.$$

6. (a) The period is p seconds.

(b) The sketches are shown in Figure 1.

(c) $\omega_0 = 2\pi/p$ radians/second.

(d) We have $Y_0 = 1/p\int_0^p y(t)dt$. So $Y_0 = 1/2$ if $p = 2$, and $Y_0 = 1/4$ if $p = 4$.

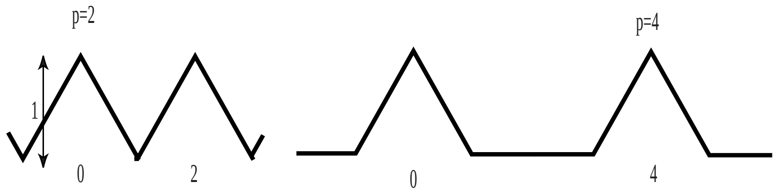


Figure 1: The graph of y for problem 6

7. (a) The zero-state impulse response is:

$$\forall n, \quad h(n) = y(n) = \begin{cases} d, & n = 0 \\ c^{\prime} A^{n-1} b, & n \geq 0 \end{cases}$$

(b) Substituting above and using the hint gives: $h(0) = 1$ and for $n \geq 1$

$$h(n) = [0 \quad 1] \left[\begin{array}{cc} \cos((n-1)\theta) & \sin((n-1)\theta) \\ -\sin((n-1)\theta) & \cos((n-1)\theta) \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = -\sin((n-1)\theta).$$

8. (a) A finite image is a signal whose domain is a subset

$$[a, b] \times [c, d] \subset \textit{Reals} \times \textit{Reals}$$

for some finite a, b, c, d where $a < b$ and $c < d$. A periodic image will have to have an infinite domain. The signal

$$\textit{Image}\colon \textit{Reals} \times \textit{Reals} \to \textit{Intensity}$$

is periodic with horizontal period $p_h \in \textit{Reals}$ and vertical period $p_v \in \textit{Reals}$ if

$$\forall\, x, y \in \textit{Reals}, \quad \textit{Image}(x, y) = \textit{Image}(x + p_h, y) = \textit{Image}(x, y + p_v).$$

As a consequence,

$$\textit{Image}(x, y) = \textit{Image}(x + kp_h, y + mp_v)$$

for all integers k, m.

(b) A finite discrete-time signal is a signal whose domain is a some finite subset

$$[a, b] \subset \textit{Ints},$$

where $a, b \in \textit{Ints}$ and $a < b$. A discrete-time signal $y\colon \textit{Ints} \to \textit{Reals}$ is periodic with period $p \in \textit{Ints}$ if

$$\forall\, n \in \textit{Ints}, \quad y(n) = y(n + p).$$

As a consequence, $y(n) = y(n + kp)$, for all integers k.

9. (a) A radio wave travels at the speed of light, c m/s. Furthermore, as it travels, the wave gets attenuated (because its power spreads in three dimensional space). So if x is the transmitted radio wave, the radio wave received at a distance l meters away at time t will be $y(t) = ax(t - l/c)$, where a is the attenuation $(a < 1)$.

(b) If there were 10 reflected paths (one direct, 9 reflected), the received signal would be $\forall\, t \in$ *Reals*,

$$y(t) = A \sum_{k=0}^{9} \alpha_k \cos(2\pi f(t - l_k/c))$$

where l_k is the length of the k-th path and is the attenuation along that path.

(c) The phase shift along the k-th path is

$$\phi_k = 2\pi f l_k / c,$$

which has units of radians. Notice that the wavelength is $\lambda = c/f$, so

$$\phi_k = 2\pi l_k / \lambda,$$

(d) Since $\phi_k - \phi_0 = 2\pi f(l_k - l_0)$,

$$\Phi = 2\pi L f / c.$$

(e) In this case,

$$y(t) = A\alpha_0 \cos(2\pi f t - \phi_0) + A\alpha_1 \cos(2\pi f t - \phi_0 - \pi).$$

Note that in general $\cos(\theta - \pi) = -\cos(\theta)$, so

$$y(t) = A\cos(2\pi f t - \phi_0)(\alpha_0 - \alpha_1).$$

So the amplitude of the direct signal is reduced by the full amount of the amplitude of the reflected signal. The reflected signal is said to be 180 degrees out of phase.

(f) Using the result of (d), to get $\Phi \le \pi/10$ we need

$$2\pi L f / c \le 10.$$

Since $L \le 500$, we need to constrain f so that

$$2\pi 500 f / c \le 10.$$

Using $c = 3 \times 10^8$, we need

$$f \le 3 \times 10^4.$$

So if we can use a radio frequency below 30 kHz, then we are assured of not having much destructive interference. Unfortunately, there is not much bandwidth below 30 kHz, and it is impractical to build mobile radios at these frequencies anyway because of the huge antennas that are required. Thus, mobile radio systems have to contend with destructive multipath interference.

(g) A frequency interferes destructively if

$$\phi_1 - \phi_0 = 2\pi f L / c = (2n + 1)\pi.$$

for some integer n. I.e.,

$$L = (2n + 1)c/2f = (2n + 1)\lambda/2.$$

Thus, we get destructive interference if L equals an odd multiple of half the wavelength.

10. Denote the square wave by $s_p\colon \mathit{Reals} \to \mathit{Reals}$. Assume the periodic image is a function

$$\mathit{Image}\colon \mathit{Reals} \times \mathit{Reals} \to \mathit{Reals}$$

given by $\forall\ (x, y) \in \mathit{Reals} \times \mathit{Reals}$,

$$\mathit{Image}(x, y) = s_h(x)s_v(y).$$

Thus, given that

$$s_p(t) = A_0 + \sum_{k=0}^{\infty} A_k \cos(2\pi kt/p + \phi_k)$$

we have

$$\mathit{Image}(x, y) = [A_0 + \sum_{k=0}^{\infty} A_k \cos(2\pi kx/h + \phi_k)][A_0 + \sum_{k=0}^{\infty} A_k \cos(2\pi ky/v + \phi_k)].$$

11. Note that

$$s(x) = \mathit{Im}\{e^{(-x+i2\pi x)}\} = e^{-x} \sin(2\pi x)$$

which is plotted in figure 2. This plot was generated by the following Matlab code:

```
x = [-1:0.01:1];
s = exp(-x).*sin(2*pi*x);
plot(x,s)
```

Obviously, on a midterm exam you will not be able to use Matlab to create this plot. You should be able to do this by hand.

12. (a) We need to show that for all $x \in [\mathit{Reals} \to \mathit{Reals}]$ and $t \in \mathit{Reals}$,

$$\mathit{Squarer}(x))(t) = f(x(t))$$

for some function $f\colon \mathit{Reals} \to \mathit{Reals}$. Let f be such that for all $u \in \mathit{Reals}$, $f(u) = u^2$. Then

$$f(x(t)) = x^2(t) = \mathit{Squarer}(x))(t).$$

Hence the function is memoryless.

(b) To show that the system is not linear, it is sufficient to show that for some $a \in \mathit{Reals}$ and $x \in [\mathit{Reals} \to \mathit{Reals}]$,

$$\mathit{Squarer}(ax) \neq a\mathit{Squarer}(x).$$

To show this, we can find any $t \in \mathit{Reals}$ such that

$$(\mathit{Squarer}(ax))(t) \neq a(\mathit{Squarer}(x)(t)).$$

The left side equals $a^2x^2(t)$, while the right side is $ax^2(t)$. These are not equal if, for example, $a = 2$ and $x(t) \neq 0$.

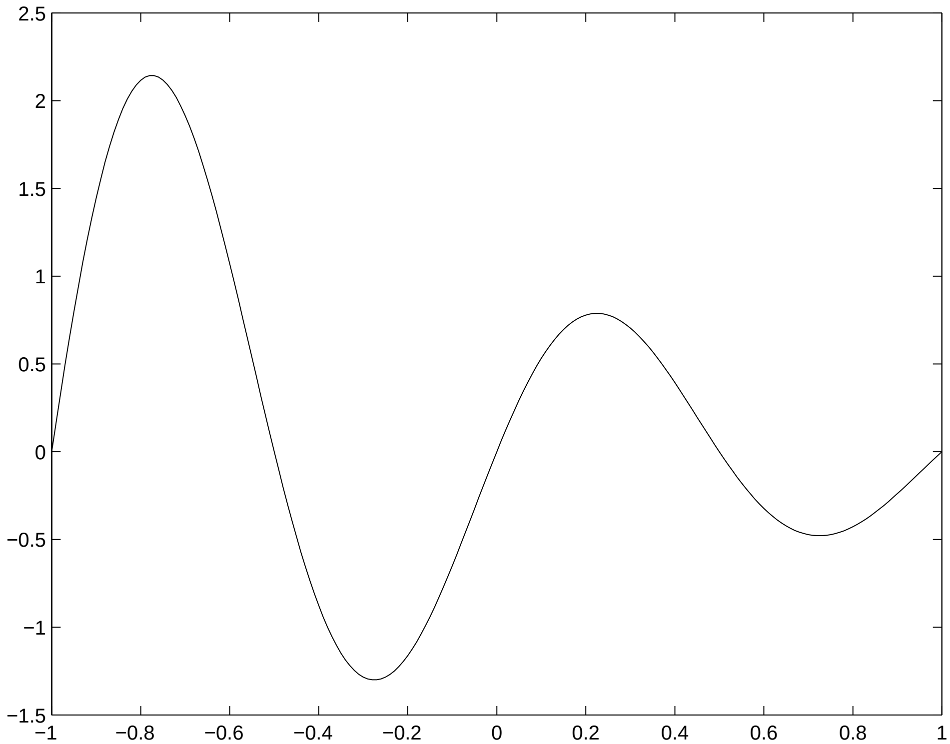


Figure 2: Solution to problem 11.

(c) We need to show that

$$Squarer \circ D_\tau = D_\tau \circ Squarer.$$

Both sides define the same function f given by

$$\forall\, x \in [Reals \to Reals], t \in Reals, \quad (f(x))(t) = x^2(t - \tau).$$

(d) Observe that

$$\begin{aligned} y(t) &= \cos^2(\omega t) \\ &= (0.5(e^{i\omega t} + e^{-i\omega t}))^2 \\ &= 0.25(e^{i2\omega t} + 1 + 1 + e^{-ie\omega t})^2 \\ &= 0.5 + 0.5\cos(2\omega t), \end{aligned}$$

from which we see that the output contains a constant term and a sinusoidal term at frequency 2ω.

13. (a) Yes, it is linear. Let $y_1 = TimeScale(x_1)$, and $y_2 = TimeScale(x_2)$. I.e.,

$$y_1(t) = x_1(2t) \quad \text{and} \quad y_2(t) = x_2(2t).$$

Let $y = TimeScale(ax_1 + bx_2)$. I.e.,

$$y(t) = ax_1(2t) + bx_2(2t).$$

Observe from this latter equation that

$$y = a \cdot TimeScale(x_1) + b \cdot TimeScale(x_2),$$

so superposition applies.

(b) No, it is not time invariant. Let

$$y_1 = (D_\tau \circ TimeScale)(x)$$

and

$$y_2 = (TimeScale \circ D_\tau)(x).$$

I.e.,

$$y_1(t) = x(2t - \tau)$$

and

$$y_2(t) = x(2t - 2\tau).$$

These are not equal for all x, so the system is not time invariant.

14. Note that

$$x(n) = e^{i\omega n}$$

where $\omega = 0$. Therefore, $y = Filter(x)$ is given by

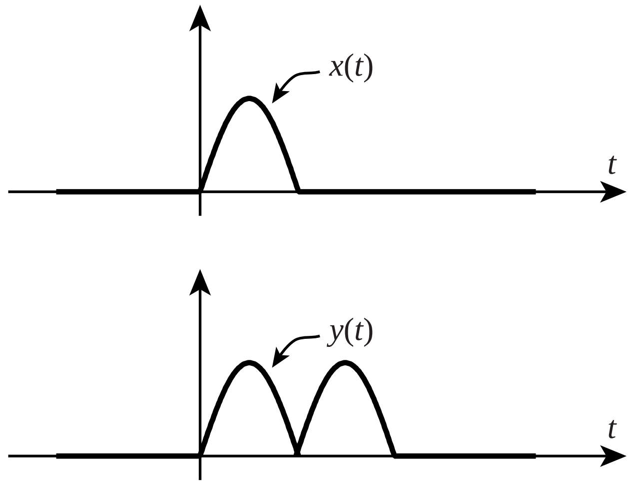
$$y(n) = H(\omega)e^{i\omega n} = H(0) \cdot 1 = |\sin(0)| = 0.$$

Hence,

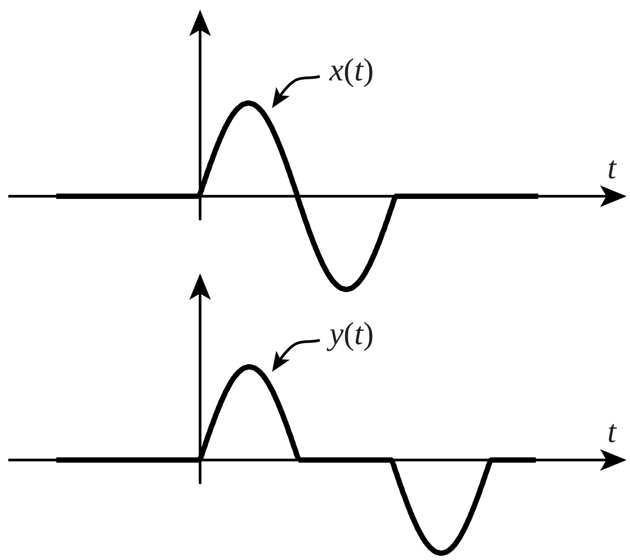
$$y(n) = 0$$

for all $n \in$ *Ints*.

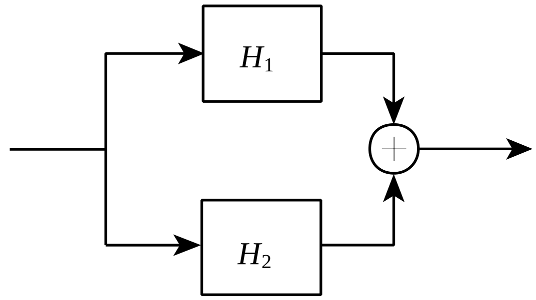
15. (a) The sketches are shown below:



(b) The sketches are shown below:



16. The system is shown below:


H_1
H_2