UNIVERSITY OF CALIFORNIA

College of Engineering
Department of Electrical Engineering and
Computer Sciences

Professor Tse Spring 2005

EECS 20n — Final Exam Solutions

[5 pts.] Problem 1

$$y(n) = \frac{1}{4}y(n-1) + x(n) + \frac{1}{2}x(n-1)$$

This is a linear constant coefficient system that is initially at rest

$\rightarrow$ LTI.

[15 pts.] Question 2

(a) (10 pts.) $$q(n) = \underbrace{\frac{1}{2}e^{i\frac{2\pi}{3}n} + \frac{1}{2}e^{-i\frac{2\pi}{3}n}}_{\text{period 3}} + \underbrace{e^{i\pi n}}_{\text{period 2}}$$

Overall period = 6 $\boxed{\omega_0 = \frac{2\pi}{6}}$

$X_2 = X_{-2} = \frac{1}{2}$

$X_3 = 1$ $\boxed{\text{for } -2 \le k \le 3}$

$X_k = 0$ otherwise

(b) (5 pts.) $$v(t) = \underbrace{\cos(\pi t)}_{\text{period 2}} + \underbrace{\cos(t)}_{\text{period } 2\pi}$$

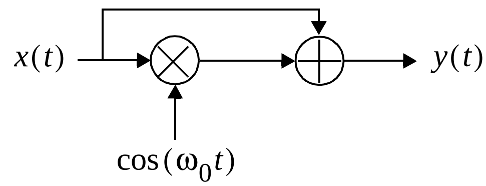
Overall period $p = 2m = 2\pi n$ where m, n are integers.

No such m, n exists because 2 is rational and 2π is irrational,

$\therefore$ not periodic.

[10 pts.] Question 3

(a) (5 pts.) Yes, the system F can be linear.



(b) (5 pts.) The system F cannot be time-invariant, because new frequencies have been created that did not exist in $X(\omega)$.

[15 pts.] Question 4

(a) (7 pts.) Reading the signals off the diagram, we have:

$$y(n) = -5y(n-1) - y(n-1) + x(n-1) + x(n)$$

Written alternatively, the LCCDE describing the system above is:

$$y(n) + 5y(n-1) + y(n-1) = x(n) + x(n-1)$$

(b) (8 pts.) We note that if $s_i(n)$ is the output of a delay block, then the input to the delay block must be $s_i(n+1)$, as shown below:

$$s_i(n+1) \longrightarrow \boxed{D} \longrightarrow s_i(n) \qquad i = 1, 2$$

Accordingly, we can label the original delay-adder-gain block diagram with $s_1(n+1)$ and $s_2(n+1)$.

We can now read off the diagram the expressions for $s_1(n+1)$, $s_2(n+1)$, and $y(n)$.

$$s_1(n+1) = s_2(n) - 5y(n) + x(n) \} \Rightarrow s_1(n+1) = -5s_1(n) + s_2(n) - 4x(n)$$

$$\left.\begin{aligned} y(n) &= s_1(n) + x(n) \\ s_2(n+1) &= -y(n) \end{aligned}\right\} \Rightarrow s_2(n+1) = -s_1(n) - x(n)$$

Hence, the state-space equations are:

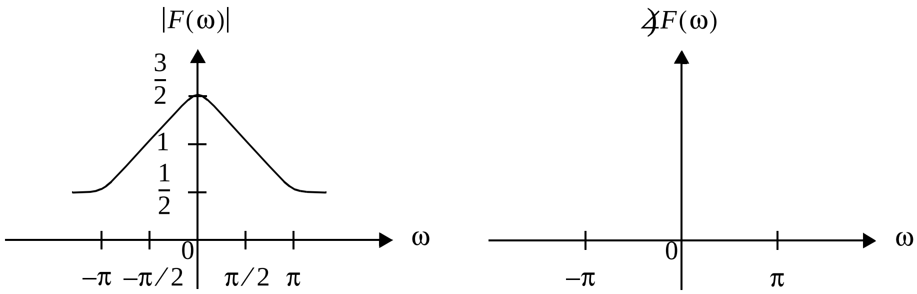
$$\begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}_{b} x(n)$$

$$y(n) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c^T} \begin{bmatrix} s_1(n) \\ s_{2(n)} \end{bmatrix} + \underset{\substack{\uparrow \\ d}}{1} \cdot x(n)$$

[25 pts.] Question 5

(a) (7 pts.) $F(\omega) = \sum_n f(n)e^{-i\omega n} = \frac{1}{4}e^{i\omega} + 1 + \frac{1}{4}e^{-i\omega} = 1 + \frac{1}{2}\cos\omega$

$$\left.\begin{array}{l} F(\omega) \in \mathbb{R} \\ F(\omega) > 0 \end{array}\right\} \Rightarrow \begin{array}{l} F(\omega) = |F(\omega)| = 1 + \frac{1}{2}\cos\omega \\ \measuredangle F(\omega) = 0 \text{ for a positive real quantity} \end{array}$$

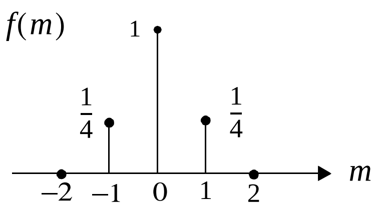


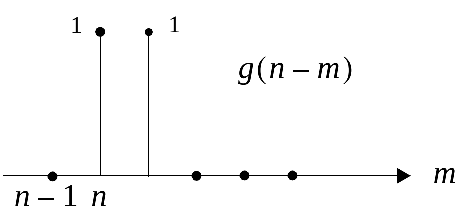
Note: $F(\omega)$, $|F(\omega)|$, and $\measuredangle F(\omega)$ are periodic and repeat outside the $(-\pi, \pi)$ interval.

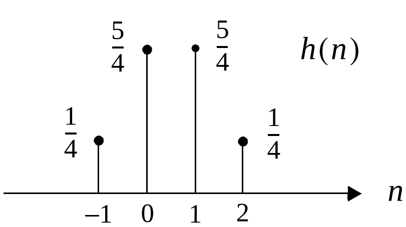
(b) (8 pts.) $h = f * g$ for a cascade interconnection.

$$h(n) = \sum_m f(m)g(n-m)$$

$$h(n) = \begin{cases} \frac{1}{4} & n = -1, 2 \\ \frac{5}{4} & n = 0, 1 \\ 0 & \text{elsewhere} \end{cases}$$







Sanity Check:

$f(n)$ has a region of support of length $N = 3$.

$g(n)$ has a region of support of length $M = 2$.

$h(n)$ has a region of support of length $N + M - 1 = 4$.

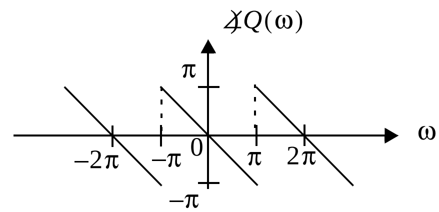
(c) (10 pts.) Recognize that $g(n) = f(n-1)$ of Part (a). Hence,

$$Q(\omega) = F(\omega)e^{-i\omega} = \left(1 + \frac{1}{2}\cos\omega\right)e^{-i\omega} \Rightarrow$$

$$|Q(\omega)| = |F(\omega)| = 1 + \frac{1}{2}\cos\omega \qquad \measuredangle Q(\omega) = -\omega \qquad -\pi < \omega < \pi$$

Periodic with period 2π

We plot only the phase here:



$$r(n) = \cos\left(\frac{7\pi n}{3} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi n}{3} + \frac{\pi}{3}\right)$$

$$\omega_0 = \frac{\pi}{3}; \; \theta = \frac{\pi}{3}$$

For an LTI system

$$\cos(\omega_0 n + \theta) \longrightarrow \boxed{\begin{array}{c} Q(\omega) \\ \text{LTI} \end{array}} \longrightarrow |Q(\omega_0)|\cos(\omega_0 n + \theta + \measuredangle Q(\omega_0))$$

This can be verified by recasting $\cos(\omega_0 n + \theta)$ in terms of complex exponentials and simplifying expressions after invoking the eigenfunction property of complex exponentials. Hence, the output $v(n)$ is:

$$v(n) = \left|Q\left(\frac{\pi}{3}\right)\right|\cos\left(\frac{\pi n}{3} + \frac{\pi}{3} + \measuredangle Q\left(\frac{\pi}{3}\right)\right) = \frac{5}{4}\cos\left(\frac{\pi n}{3} + \frac{\pi}{3} - \frac{\pi}{3}\right)$$

$$v(n) = \frac{5}{4}\cos\left(\frac{\pi n}{3}\right)$$

[15 pts.] Question 6

(a) (4 pts.) The unit of ω_0 is **radians/second**.

The unit of X_k is **volts**.

(b) (6 pts.) The unit of ω is **radians/second**.

The unit of $\mathrm{d}\omega$ is **radians/second**.

The unit of $X(\omega)$ is: $\frac{X(\omega)\mathrm{d}\omega}{2\pi}$ has unit of volt $\Rightarrow X(\omega)$ has unit of $\frac{\mathrm{volt} \cdot \mathrm{rad}}{\mathrm{rad}/\mathrm{sec}} = \mathrm{volt} \cdot \mathrm{sec}$.

No, X_k and $X(\omega)$ do not have the same unit; X_k has the same unit as $\frac{X(\omega)\mathrm{d}\omega}{2\pi}$.

(c) (5 pts.) The unit of Ω is **radians/sample**.

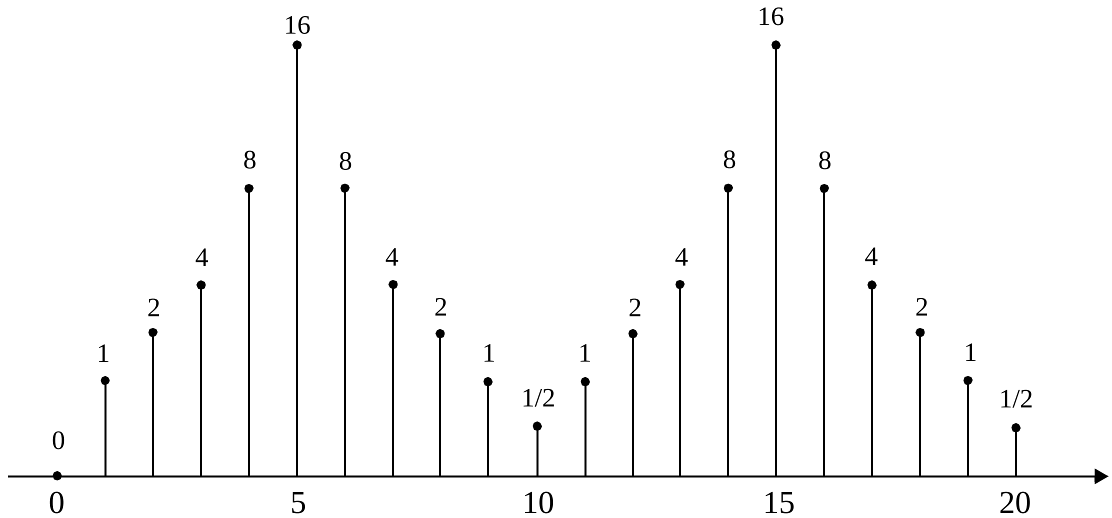
The unit of $\mathrm{d}\Omega$ is **radians/sample**.

The unit of $Y(\Omega)$ is: $\frac{Y(\Omega)\mathrm{d}\Omega}{2\pi}$ has unit of volt $\Rightarrow Y(\Omega)$ has unit of $\frac{\mathrm{volt} \cdot \mathrm{radians}}{\mathrm{radians}/\mathrm{sample}} = \mathrm{volt} \cdot \mathrm{sample}$.

No, the units of ω and Ω are not the same. ω is the frequency of a CT signal $\Rightarrow$ has unit of radians/sec. Ω is the frequency of a DT signal $\Rightarrow$ has unit of radians/sample. Again here, 2π has unit of radians.

[20 pts.] Question 7

(a) (5 pts.)



(b) (9 pts.) No, overall state $S = \{1, 2\} \times \text{Reals}$

$$\text{state} = (\text{mode}(n), r(n))$$

$\text{mode}(n)$	$r(n)$	$\text{mode}(n+1)$	$r(n+1)$	output $y(n)$
1	$\lvert r(n)\rvert \le 10$	1	$2r(n) + x(n)$	$r(n)$
1	$\lvert r(n)\rvert > 10$	2	$\frac{1}{2}r(n) + x(n)$	$r(n)$
2	$\lvert r(n)\rvert < 1$	1	$2r(n) + x(n)$	$r(n)$
2	$\lvert r(n)\rvert \ge 1$	2	$\frac{1}{2}r(n) + x(n)$	$r(n)$

(c) (6 pts.) $\text{state}(n) = (\text{mode}(n),\ \text{time in mode 2},\ r(n))$

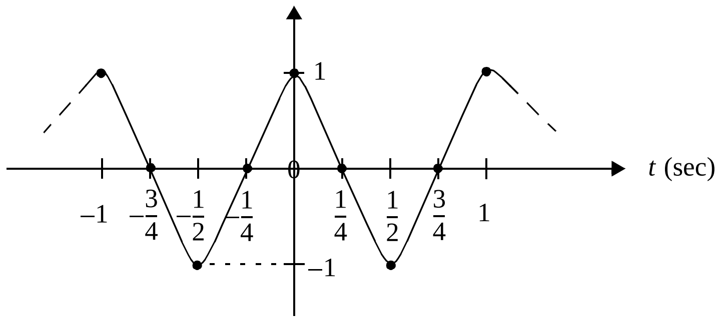
$$S = \{1, 2\} \times \{0, 1, 2\} \times \text{Reals}$$

[10 pts.] Question 8

(a) (5 pts.) We sample four times every second, at equally-spaced points in time:

$$T - 0.25 \text{ sec } \Rightarrow f_s = 4 \text{ Hz} \qquad (\omega_s = 8\pi \text{ rad/sec})$$

where s is the sampling frequency.



(b) (5 pts.) $f_s - f_2$ will appear as $f_1 \Rightarrow 4 = f_2 = f_1 = 1 \Rightarrow f_2 = 3$ Hz.

Verify: $y(t) = \cos(6\pi t)$ evaluated at $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

Second points:

$$y(0) = \cos(0) = 1 = x(0);\ y\left(\frac{1}{4}\right) = \cos\left(\frac{6\pi}{4}\right) = \cos\left(\frac{3\pi}{2}\right) = 0 = x\left(\frac{1}{4}\right)$$

$$y\left(\frac{1}{2}\right) = \cos\left(\frac{6\pi}{2}\right) = \cos(3\pi) = -1 = x\left(\frac{1}{2}\right);\text{ and } y\left(\frac{3}{4}\right) = \cos\left(\frac{6\pi \cdot 3}{4}\right) = \cos\left(\frac{9\pi}{2}\right) = 0$$

$$= x\left(\frac{3}{4}\right)$$

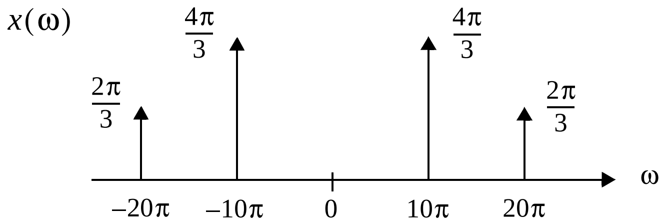
[15 pts.] Question 9

(a) (5 pts.) $\omega_0 = 10\pi$ rad/sec $\Rightarrow x_1 = \frac{2}{3} \overset{F}{\leftrightarrow} e^{i10\pi t}$; $x_{-1} = \frac{2}{3} \overset{F}{\leftrightarrow} e^{-i10\pi t}$

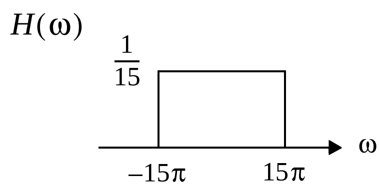
$$x_2 = \frac{1}{3} \overset{F}{\leftrightarrow} e^{i20\pi t};\quad x_{-2} = \frac{1}{3} \overset{F}{\leftrightarrow} e^{-i20\pi t}$$

$$\Rightarrow x(t) = \frac{2}{3}(e^{i10\pi t} + e^{-i10\pi t}) + \frac{1}{3}(e^{i20\pi t} + e^{-i20\pi t}) \Rightarrow$$

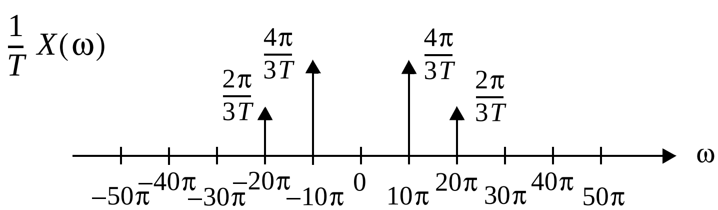
$$x(t) = \frac{4}{3}\cos(10\pi t) + \frac{2}{3}\cos(20\pi t)$$

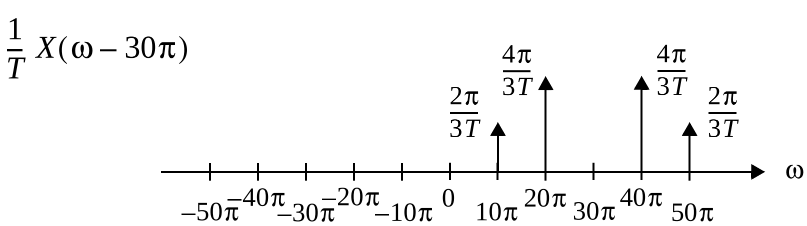


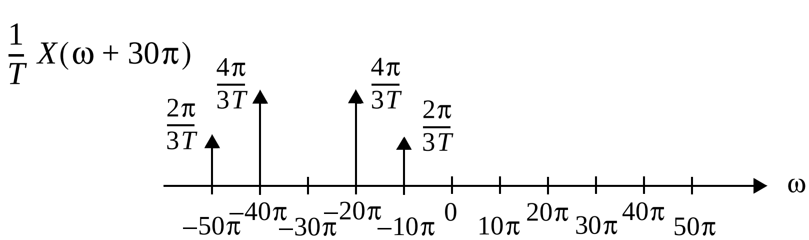
(b) (10 pts.) $f_s = 15$ Hz ($\omega_s = 30\pi$ rad/sec) $\Rightarrow X_p(\omega) = \frac{1}{T}\sum_k X(\omega - k\omega_s) = 15\sum_k X(\omega - 30k\pi)$



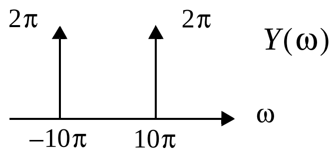
Because of the filter $H(\omega)$, all frequency content of $X_p(\omega)$ outside of the $(-15\pi, 15\pi)$ band will be suppressed. $\Rightarrow$ We need only look at $\frac{1}{T}X(\omega)$, $\frac{1}{T}X(\omega - 30\pi)$, and $\frac{1}{T}X(\omega + 30\pi)$:







The above three components [scaled spectral replica of $X(\omega)$] are added and only the frequency content in the $(-15\pi, 15\pi)$ band is retained (and scaled). Hence, the spectrum of $y(t)$ is:



$\Rightarrow$ $y(t) = 2\cos(10\pi t)$ (where 20π frequency components are aliased down to 10π).