EECS 20. Final Exam Practice Problems, Spring, 2005.

1. Consider a LTI system with $[A, b, c, d]$ representation given by:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^T = [1 \;\; 0], \quad d = 0.$$

   (a) Calculate the zero-input state response when the initial state is $s(0) = [s_1(0) \quad s_2(0)]^T$.

   (b) Calculate the (zero-state) impulse response, h.

   (c) Calculate the response $y(n), n \geq 0$ when the initial state is $s(0) = [1 \quad 1]^T$ and the input signal is $\forall n \geq 0, x(n) = \delta(n-1)$.

2. Consider the difference equation

$$y(n) - y(n-1) = x(n) - 2x(n-1).$$

   (a) Take the state at time n as $s(n) = [y(n-1), x(n-1)]^T$ and write down the $[A, b, c, d]$ representation of the system. Find its zero-state impulse response.

   (b) Implement the difference equation using two delay elements whose outputs are the two state components.

   (c) Find another implementation using only *one* delay element. Write the $[A, b, c, d]$ representation for this implementation. Find its zero-state impulse response.

   (d) Are the two impulse responses the same?

   (e) Find the frequency response directly from the difference equation and by taking the DTFT of the impulse response. Are the two frequency responses the same?

   (f) Sketch the magnitude and phase response.

3. Consider a LTI system with $[A, b, c, d]$ representation given by:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c^T = [1 \;\; 1], \quad d = 0.$$

   (a) Suppose the initial state is $s(0) = [0 \;\; 0]^T$. Find an input sequence $x(0), x(1)$ of length two such that the state at time 2 is $s(2) = [1 \;\; 1]^T$.

   (b) Suppose the initial state is $s(0) = [s_1 \quad s_2]^T$. Find an input sequence $x(0), x(1)$ such that the state at time 2 is $s(2) = [0 \;\; 0]^T$. (The input sequence will depend on $s(0)$.)

4. Two SISO systems with representations $[A_i, b_i, c_i, d_i], i = 1, 2$ are connected in cascade composition. Find an $[A, b, c, d]$ representation for the composiiton.

5. A discrete-time, causal LTI system S produces the output y given by

$$y(n) = \delta(n) + \delta(n-1) + \delta(n-2),$$

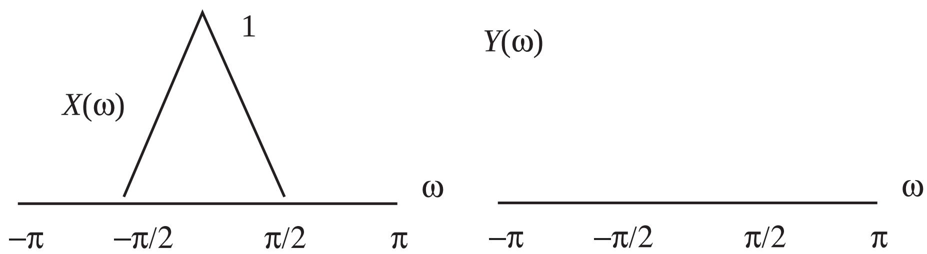


Figure 1: Signal x for problem 7.

in response to the input x given by

$$x(n) = \delta(n) + \delta(n-2).$$

Find the impulse response h of S.

6. Evaluate the convolution integral $y_i = h_i * x$ when x is the continuous-time unit step function , $x(t) = 0, t < 0; = 1, t \geq 0$, and the impulse response h_i is as given below, $i = 1, 2, 3$.

   (a) $h_1(t) = 0, t < 0; = e^{-t}, t \geq 0$.
   (b) $h_2(t) = e^{t}, t < 0; = 0, t \geq 0$.
   (c) $h_3(t) = e^{t}, t < 0; = e^{-t}, t \geq 0$.

7. This problem concerns the various Fourier transforms.

   (a) The exponential Fourier series of the signal x,

   $$x(t) = \cos(2\pi t) + \sin(3\pi t),$$

   (b) The Fourier transform of the signal z,

   $$z(t) = e^{-t}, t \geq 0; = 0, t < 0,$$

   and the Fourier transform of the signal y,

   $$y(t) = z(t)e^{i\omega_0 t},$$

   (in which z is as above)

   (c) Suppose the DTFT X of a signal $x : \mathcal{I}nts \to \mathcal{C}omplex$ is as shown in figure 1.

      i. Prove that the signal x is real-valued.

      ii. Suppose the signal y is constructed by: $y(k) = x(k/2)$, if k is even; and $y(k) = 0$, if k is odd. What is the DTFT Y of y in terms of X, and sketch Y above.

8. Find the DTFT of the signal

$$\forall n \in \mathcal{I}nts, x(n) = (0.5)^{|n|}$$

9. Recall the inverse DTFT formula

$$\forall n, \quad x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{i\omega n} d\omega.$$

   (a) Use this formula to guess and verify the DTFT of the signal $x(n) = e^{i\omega_0 n}$, where $0 \leq \omega_0 < 2\pi$.

   (b) What is the DTFT of the signal $y(n) = \cos(\omega_0 n)$ for $0 \leq \omega_0 < 2\pi$.

   (c) What is the DTFT of the signal $z(n) = e^{i\omega_0 n}$ for $\omega_0 = 2\pi + \pi/4$. Note: $\omega_0 > 2\pi$.

10. In the figure 2 on the next page, the left column shows three continuous-time signals, x, p, y.

   (a) Write down expressions for the corresponding Fourier Transforms X, P, Y.

$$\begin{aligned} X(\omega) &= \\ P(\omega) &= \\ Y(\omega) &= \end{aligned}$$

   What is the unit of ω?

   (b) Plot these Fourier Transforms in the column on the right. Mark the values at $\omega = 0$. Also, on the ω-axis, indicate the frequencies where the Fourier Transform is not zero.

   (c) Suppose the signal y is sampled every 0.01 s, i.e. the sampling frequency is 100 Hz. The discrete-time sampled signal is called z. Write down an expression for the DTFT Z of z in terms of Y.

$$Z(\omega) =$$

   What is the unit of ω?

   (d) Sketch a plot of Z in the figure.

11. The bandwidth of a continuous time signal x with FT X is by definition the smallest frequency ω_B such that $X(\omega) = 0$ for $|\omega| > \omega_B$.

   (a) What is the bandwidth of the signals: $\forall t \in \mathcal{R}eals$,

$$x_k(t) = \cos(10k\pi t), k = 1, 2, 3; \quad x_4(t) = x_1(t) + x_2(t) + x_3(t).$$

   (b) What is the FT of $x_k, k = 1, \cdots, 4$?

   (c) Suppose x_k is sampled at frequency $\omega_s = 30\pi$ rad/sec. Find a simple expression for the sampled signal y_k.

   (d) Find signals $z_k : \mathcal{R}eals \to \mathcal{R}eals$ such that (i) the bandwidth of z_k is smaller than 15π rad/sec, which is one-half the sampling frequency; and (ii) if z_k is sampled at frequency ω_s it also yields the signal y_k.

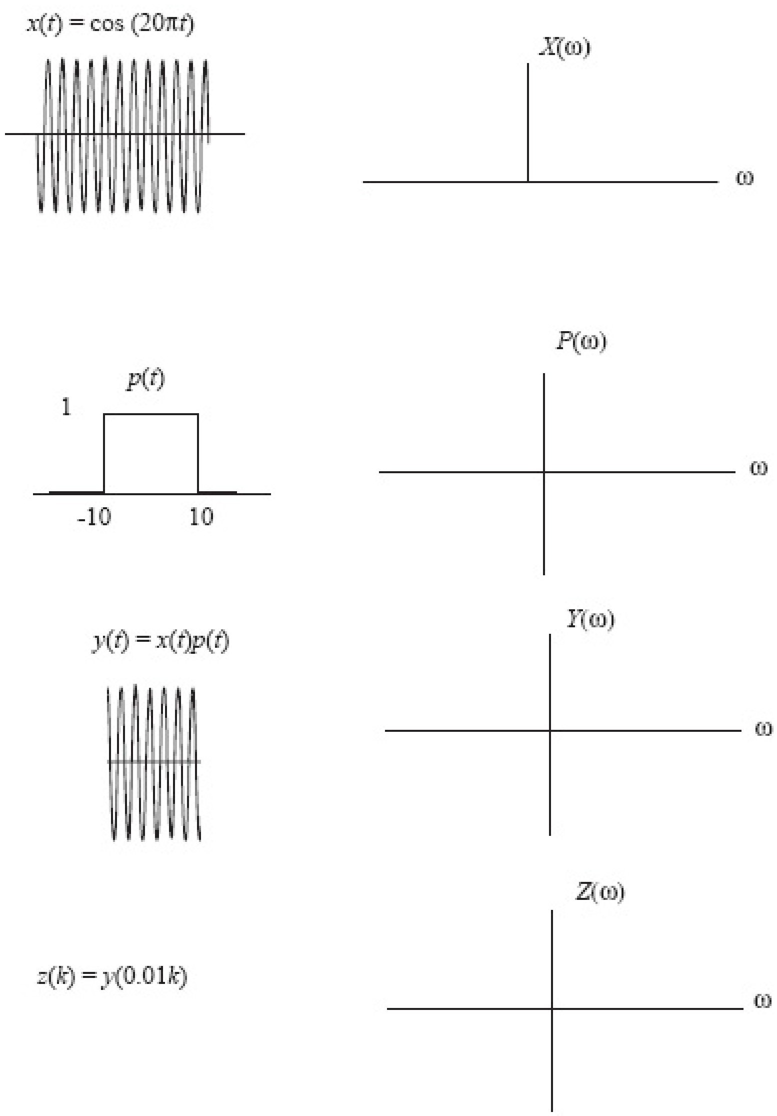


Figure 2: Signals for problem 10.

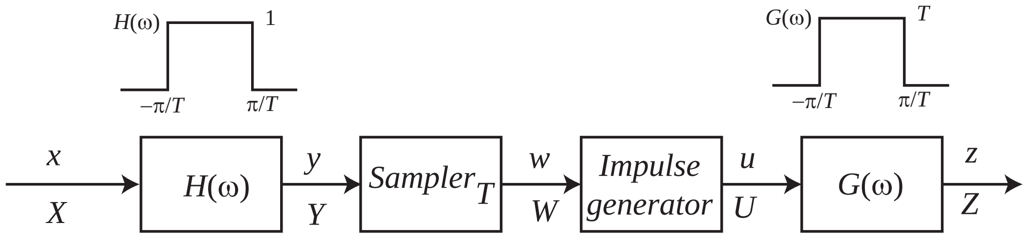


Figure 3: Setup for problem 13.

12. We sub-sample a discrete-time signal x to yield another discrete-time signal y, where $y(n) = x(n)$ if n is a multiple of L and $y(n) = 0$ otherwise. Here, we are sampling one out of every L samples of x. Compute the DTFT of y in terms of the DTFT of x. Under what condition on the DTFT of x would one be able to reconstruct x from y? How would you do the reconstruction? (Hint: mimic what we did for the CT to DT sampling.)

13. Consider the setup of figure 3. The filters H, G are as shown; the sampling period is T seconds.

    (a) Express w, u in terms of y and W, U in terms of Y.

    (b) Express Z in terms of X.

    (c) Determine y and z for $T = 0.1s$ and $\forall t, x(t) = \sin(25\pi t) + \sin(5\pi t)$.

    (d) Suppose in this setup H is changed to $\forall \omega, H(\omega) = 1$. Take T, x as above, and determine z.

14. The figure 4 below is an incomplete description of a controller. When someone presses the *open* button, the output is turned *on* and 15 sec later it is turned *off*. If the *open* button is pressed before the output is *off* the output stays *on* for 15 sec beyond the last time the *open* button was pressed. If someone presses *close* while the output is *on*, it is immediately turned *off*.

    (a) Design the guards, actions, and outputs for the transitions so as to meet this specification. Two modes are available, as shown in the diagram. However, you may use only one mode.

    (b) Sketch the output signal y when the input signal x is as shown. Mark all time instances t when y changes value.

15. a) Show cos(u+v) = cos(u)cos(v) - sin(u)sin(v) using complex numbers.

    b) Squares cbqp and acmn are erected outwardly on the sides bc and ac of the triangle abc. Let d,e be the centers of these squares. Let g be the midpoint of ab, and f be the midpoint of mp. Show eg and gd are perpendicular, by representing the points on a complex plane.

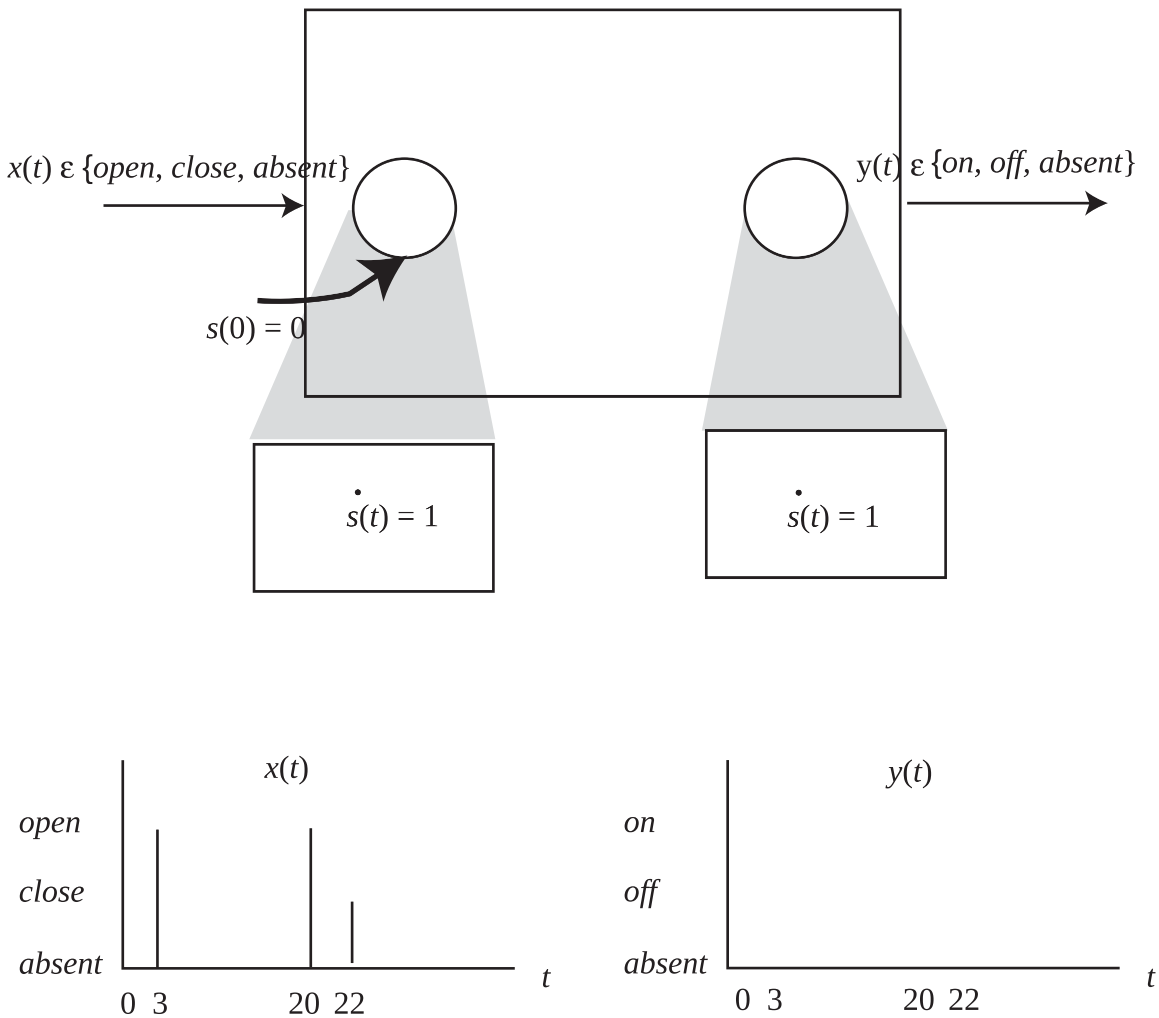
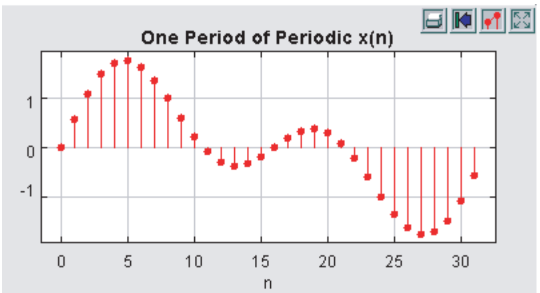
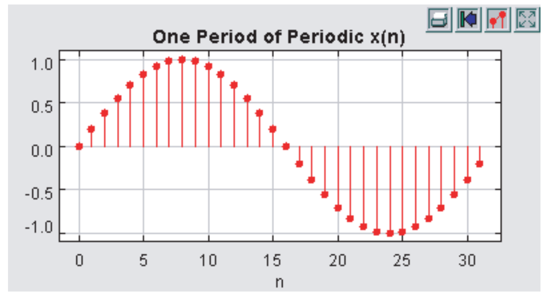


Figure 4: Figures for problem 14.

16. Suppose that x is a discrete-time signal with period $p = 32$. Below are plotted six possible such signals x. For each of these, match one of the six plots on the next page, or indicate that none matches.



matching FS:________________________



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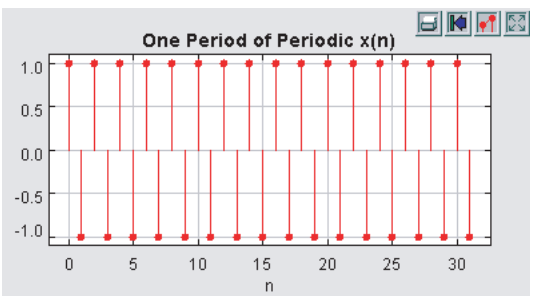

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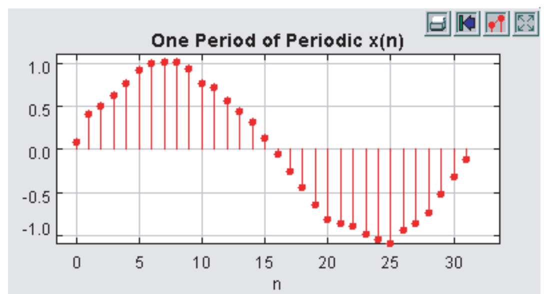

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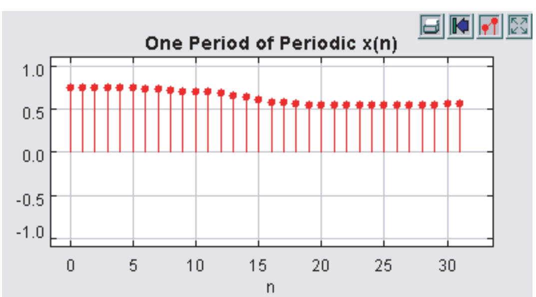

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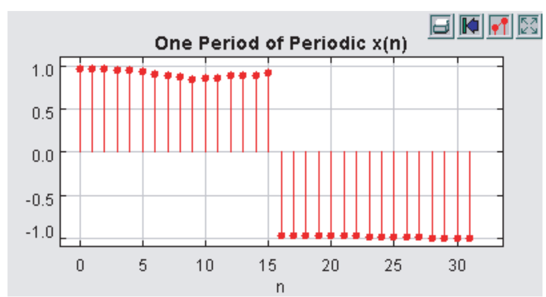

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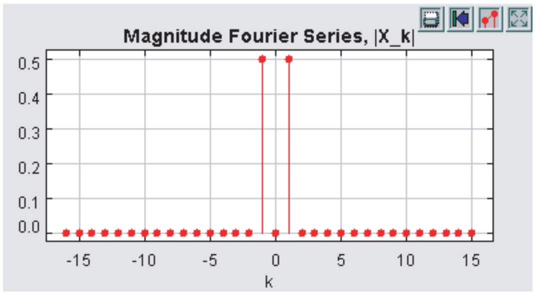
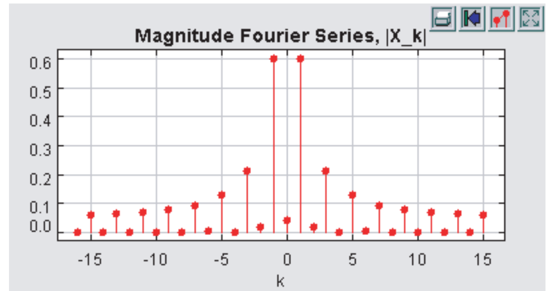
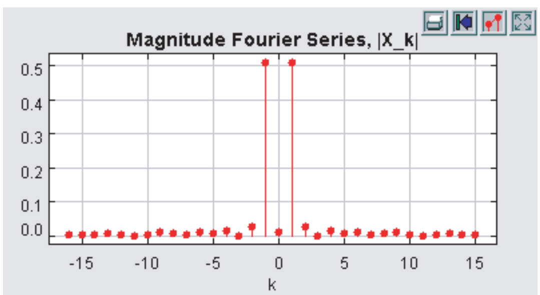
Below are six plots of the magnitudes $|X_k|$ of the Fourier series coefficients X_k of a periodic signal x with period $p = 32$, for six different such periodic signals.



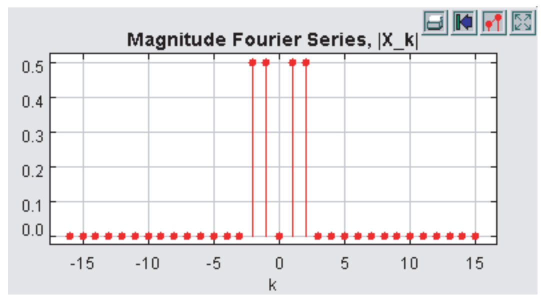
(a)



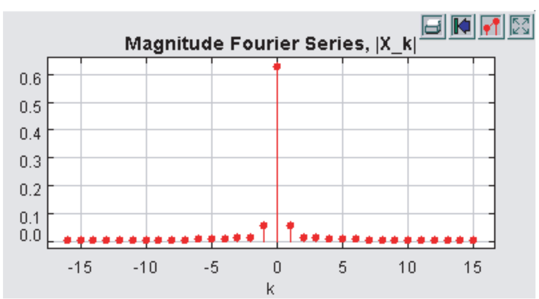
(b)



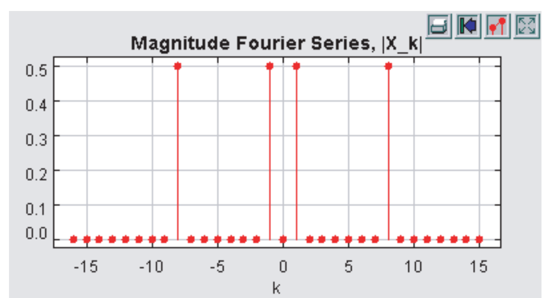
(c)



(d)



(e)



(f)

17. Consider a discrete-time function x where

$$\forall\, n \in, \quad x(n) = 1 + \cos(\pi n/4) + \sin(\pi n/2).$$

(a) Find the period p and fundamental frequency ω_0. Give the units.

(b) Find K and the Fourier series coefficients $A_0, \cdots, A_K$ and $\phi_1, \cdots, \phi_K$ in

$$x(n) = A_0 + \sum_{k=1}^{K} A_k \cos(k\omega_0 n + \phi_k).$$

(c) Find the Fourier series coefficients X_k in

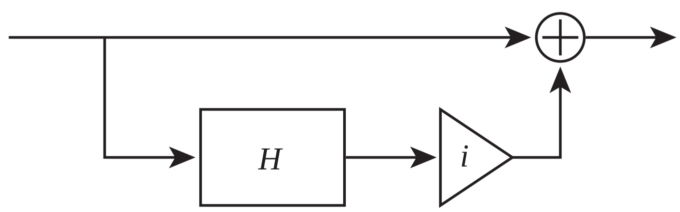
$$x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}.$$

(d) Let x be the input to an LTI system with frequency response H given by

$$\forall\, \omega \in, \quad H(\omega) = \begin{cases} -i & \text{if } 0 < \omega < \pi \\ 0 & \text{if } \omega = 0 \text{ or } \omega = \pi \\ i & \text{if } -\pi < \omega < 0 \end{cases}$$

For other values of ω, $H(\omega)$ is determined by the periodicity of H. (Such a system is called a Hilbert filter.) Find the output.

(e) Find the frequency response H' of a new system constructed as follows,



where H is the Hilbert filter from part (d), and the triangle with an i scales its input by the imaginary number i. Note that even if the input is real-valued, the output is likely to be complex-valued. Such a system is called a Hilbert transformer; approximations to it are widely used in digital communication systems.

(f) Let the input to the system in part (e) be x. What is the output?

18. Let the continuous-time signal c given by

$$\forall\, t \in, \quad c(t) = 2\cos(\omega_c t)$$

be a carrier wave for a radio signal. Let x given by

$$\forall\, t \in, \quad x(t) = 2\cos(\omega_x t)$$

be the signal to be carried by that radio signal (that it, it is a highly simplified stand-in for, say, a voice signal). To be concrete, let $\omega_c = 2\pi \cdot 8000$ radians/second, and $\omega_x = 2\pi \cdot 400$ radians/second.

(a) Find and sketch the CTFT of y where

$$\forall\, t \in, \quad y(t) = c(t)x(t).$$

Label your sketch carefully. **Hint**: The CTFT of $e^{i\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

(b) Let y from part (a) be the input to an LTI system with frequency response H where

$$\forall\, \omega \in, \quad H(\omega) = \begin{cases} 0 & \text{if } \omega \leq 0 \\ 1 & \text{if } \omega > 0 \end{cases}$$

Find the output u.

(c) For the same u from part (b), let

$$u' = \mathcal{S}ampler_T(u),$$

where $T = 1/8000$ seconds. Find a simple expression for u'.

(d) Give the signal $z = \mathcal{I}dealInterpolator_T(u')$, where again $T = 1/8000$ seconds, and u' is from part (c).