EECS 20N: Structure and Interpretation of Signals and Systems Final S06
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FORMULAS & TABLES

Discrete Fourier Series (DFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \ ,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\sum_{k=\langle p \rangle}$ may denote $\sum_{k=0}^{p-1}$ or $\sum_{k=1}^{p}$.

Continuous-Time Fourier Series (FS) Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt \ ,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p. For example, $\int_{\langle p \rangle}$ can denote $\int_0^p$.

The DTFT and the Frequency Response of a DT LTI System Consider a real, discrete-time LTI system having impulse response $h : \mathbb{Z} \to \mathbb{R}$. If the system has a frequency response $H : \mathbb{R} \to \mathbb{C}$, it is given by

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}\,, \quad \forall \omega \in \mathbb{R}\,,$$

which is also known as the DTFT Analysis Equation. The impulse response of the system is given by

$$h(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\omega)\, e^{i\omega n}\, d\omega\,,$$

which is also known as the DTFT Synthesis Equation.

The CTFT and the Frequency Response of a CT LTI System Consider a real, continuous-time LTI system having impulse response $h : \mathbb{R} \to \mathbb{R}$. If the system has a frequency response $H : \mathbb{R} \to \mathbb{C}$, it is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt, \quad \forall \omega \in \mathbb{R}\,,$$

which is also known as the CTFT Analysis Equation. The impulse response of the system is given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, d\omega\,,$$

which is also known as the CTFT Synthesis Equation.

Duality Let the signal $x : \mathbb{R} \to \mathbb{C}$ have CTFT $X : \mathbb{R} \to \mathbb{C}$. More compactly, let

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega),$$

where $\mathcal{F}$ denotes the Fourier transform. Then

$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega).$$

Parseval-Plancherel-Rayleigh Identity Let the signals $x, y : \mathbb{R} \to \mathbb{C}$ have respective CTFTs $X, Y : \mathbb{R} \to \mathbb{C}$. Then

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, Y^*(\omega)\, d\omega\ .$$

Time domain	**Frequency domain**
$\forall\, n \in \mathbb{Z}, \quad x(n)$ is real	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = X^*(-\omega)$
$\forall\, n \in \mathbb{Z}, \quad x(n) = x^*(-n)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega)$ is real
$\forall\, n \in \mathbb{Z}, \quad y(n) = x(n-N)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = e^{-i\omega N} X(\omega)$
$\forall\, n \in \mathbb{Z}, \quad y(n) = e^{i\omega_1 n} x(n)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = X(\omega - \omega_1)$
$\forall\, n \in \mathbb{Z},$ $y(n) = \cos(\omega_1 n) x(n)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) + X(\omega + \omega_1))/2$
$\forall\, n \in \mathbb{Z},$ $y(n) = \sin(\omega_1 n) x(n)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) - X(\omega + \omega_1))/2i$
$\forall\, n \in \mathbb{Z},$ $x(n) = a x_1(n) + b x_2(n)$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = a X_1(\omega) + b X_2(\omega)$
$\forall\, n \in \mathbb{Z}, \quad y(n) = (h * x)(n)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = H(\omega) X(\omega)$
$\forall\, n \in \mathbb{Z}, \quad y(n) = x(n) p(n)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = \frac{1}{2\pi} \int\limits_0^{2\pi} X(\Omega) P(\omega - \Omega) d\Omega$
$\forall\, n \in \mathbb{Z},$ $y(n) =$ $\begin{cases} x(n/N) & n \text{ is a multiple of } N \\ 0 & \text{otherwise} \end{cases}$	$\forall\, \omega \in \mathbb{Z},$ $Y(\omega) = X(N\omega)$

Table 1: Properties of the DTFT.

Signal	**DTFT**
$\forall\, n \in \mathbb{Z}, \quad x(n) = \delta(n)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = 1$
$\forall\, n \in \mathbb{Z},$ $x(n) = \delta(n - N)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = e^{-i\omega N}$
$\forall\, n \in \mathbb{Z}, \quad x(n) = 1$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$
$\forall\, n \in \mathbb{Z},$ $x(n) = a^n u(n), \quad \lvert a \rvert < 1$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{1}{1 - ae^{-i\omega}}$
$\forall\, n \in \mathbb{Z},$ $x(n) = \begin{cases} 1 & \text{if } \lvert n \rvert \leq M \\ 0 & \text{otherwise} \end{cases}$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{\sin(\omega(M + 0.5))}{\sin(\omega/2)}$
$\forall\, n \in \mathbb{Z},$ $x(n) = \frac{\sin(Wn)}{\pi n}, \quad 0 < W < \pi$	$\forall\, \omega \in [-\pi, \pi],$ $X(\omega) = \begin{cases} 1 & \text{if } \lvert \omega \rvert \leq W \\ 0 & \text{otherwise} \end{cases}$

Table 2: Discrete time Fourier transforms of key signals. The function u is the unit step.

Time domain	**Frequency domain**
$\forall\, t \in \mathbb{R}, \quad x(t)$ is real	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = X^*(-\omega)$
$\forall\, t \in \mathbb{R}, \quad x(t) = x^*(-t)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega)$ is real
$\forall\, t \in \mathbb{R}, \quad y(t) = x(t-T)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = e^{-i\omega T}\, X(\omega)$
$\forall\, t \in \mathbb{R}, \quad y(t) = e^{i\omega_1 t} x(t)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = X(\omega - \omega_1)$
$\forall\, t \in \mathbb{R},$ $y(t) = \cos(\omega_1 t) x(t)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) + X(\omega + \omega_1))/2$
$\forall\, t \in \mathbb{R},$ $y(t) = \sin(\omega_1 t) x(t)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) - X(\omega + \omega_1))/2i$
$\forall\, t \in \mathbb{R},$ $x(t) = a x_1(t) + b x_2(t)$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = a X_1(\omega) + b X_2(\omega)$
$\forall\, t \in \mathbb{R}, \quad y(t) = (h * x)(t)$	$\forall\, \omega \in \mathbb{R}, \quad Y(\omega) = H(\omega) X(\omega)$
$\forall\, t \in \mathbb{R}, \quad y(t) = x(t) p(t)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(\Omega) P(\omega - \Omega) d\Omega$
$\forall\, t \in \mathbb{R},$ $y(t) = x(at)$	$\forall\, \omega \in \mathbb{R},$ $Y(\omega) = \frac{1}{\lvert a \rvert}\, X(\omega/a)$

Table 3: Properties of the CTFT.

Signal	CTFT
$\forall\, t \in \mathbb{R}, \quad x(t) = \delta(t)$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = 1$
$\forall\, t \in \mathbb{R}, \quad x(t) = \delta(t-\tau),\ \tau \in \mathbb{R}$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = e^{-i\omega\tau}$
$\forall\, t \in \mathbb{R}, \quad x(t) = 1$	$\forall\, \omega \in \mathbb{R}, \quad X(\omega) = 2\pi\delta(\omega)$
$\forall\, t \in \mathbb{R},$ $x(t) = e^{-at}\, u(t), \quad a > 0$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{1}{a + i\omega}$
$\forall\, t \in \mathbb{R},$ $x(t) = e^{-a\lvert t\rvert}, \quad a > 0$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{2a}{a^2 + \omega^2}$
$\forall\, t \in \mathbb{R},$ $x(t) = \text{sgn}(t), \forall t$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{2}{i\omega}$
$\forall\, t \in \mathbb{R},$ $x(t) = \begin{cases} \pi/a & \text{if } \lvert t\rvert \leq a \\ 0 & \text{otherwise} \end{cases}$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \frac{2\pi \sin(a\omega)}{a\omega}$
$\forall\, t \in \mathbb{R},$ $x(t) = \frac{\sin(\pi t/T)}{\pi t/T},$	$\forall\, \omega \in \mathbb{R},$ $X(\omega) = \begin{cases} T & \text{if } \lvert\omega\rvert \leq \pi/T \\ 0 & \text{otherwise} \end{cases}$

Table 4: Continuous time Fourier transforms of key signals. The function u is the unit step.