1. **30 points** Consider an LTI system with impulse response h given by

$$\forall\, n \in \textit{Ints}, \quad h(n) = \delta(n) + \delta(n-1),$$

where δ is the Kronecker delta function. Let H denote the frequency response of this system. Suppose that two copies of this system are connected in cascade, meaning that the output of the first one feeds the input of the second.

(a) Find the impulse response g of the cascade system.

(b) Give the frequency response G of the cascade system in terms of H.

(c) Find H and G as functions of ω, frequency in radians/sample.

Solution:

(a) If the input to the first system is δ, the Kronecker delta function, then the output will be h, the impulse response. But h just consists of two delta functions, one of them delayed. By linearity and time invariance, the output will therefore be $y = h + \Delta_1 h$, or

$$\forall\, n \in \textit{Ints}, \quad y(n) = \delta(n) + \delta(n-1) + \delta(n-1) + \delta(n-2)$$

or

$$y(n) = \delta(n) + 2\delta(n-1) + \delta(n-2).$$

(b) $$G = H^2$$

(c) $$\forall\, \omega \in \textit{Reals}, \quad H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = 1 + e^{-j\omega},$$

using the sifting property of the delta function.

$$G(\omega) = (1 + e^{-j\omega})^2.$$

2. **30 points** Consider an LTI system L with impulse response h given by

$$\forall\, n \in \textit{Ints}, \quad h(n) = \delta(n+2) + \delta(n-2),$$

where δ is the Kronecker delta function.

(a) Find the frequency response H as a function of ω, frequency in radians per sample.

(b) Find one value of ω such that if the input is the signal x where

$$\forall\, n \in \textit{Ints}, \quad x(n) = \cos(\omega n),$$

then the output y will be zero, i.e.

$$\forall\, n \in \textit{Ints}, \quad y(n) = 0.$$

(c) Consider another filter with impulse response

$$h_2(n) = \delta(n) + \delta(n-4).$$

Show that you can use the result of part (a) to quickly and easily find the frequency response of this version. (It is important the you make the connection with part (a), not just that you find the frequency response of the new filter).

Solution:

(a) $$\forall\, \omega \in \mathit{Reals}, \quad H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = e^{j2\omega} + e^{-j2\omega} = 2\cos(2\omega).$$

(b) $\omega = \pi/4$ works because

$$\cos(\omega n) = (e^{j\omega} + e^{-j\omega})/2$$

and $H(\omega) = H(-\omega) = 2\cos(\pi/2) = 0$. $\omega = 3\pi/4$ works also, as do negatives of these frequencies, and any offset by 2π.

(c) The impulse response of this new system is just

$$h_2 = \Delta_2 \circ h$$

so the system can be constructed as a cascade of a delay and h. Thus, its frequency response is the product of the frequency response of the delay and H, or

$$H_2(\omega) = H(\omega)e^{-j2\omega} = 2\cos(2\omega)e^{-j2\omega}.$$

3. **20 points** Let x be a continuous-time signal given by

$$\forall\, t \in \mathit{Reals}, \quad x(t) = \sin(2\pi 3000 t) + \sin(2\pi 5000 t).$$

(a) For what values of T will it be true that

$$x = \mathit{IdealInterpolator}_T(\mathit{Sampler}_T(x))?$$

(b) Find the signal y given by

$$y = \mathit{IdealInterpolator}_T(\mathit{Sampler}_T(x))$$

where $T = 1/8000$ is the sampling interval. **Hint:** Do not use the sinc function definition of $\mathit{IdealInterpolator}_T$. Consider instead the aliases of the two sinusoidal components.

Solution:

(a) For all $T < 1/10000$ seconds this will be true.

(b) The sine at 5000 Hz will be aliased to -3000 Hz, which will be equivalent the negative of a sine at 3000 Hz (the sine function is antisymmetric). Thus, the alias will cancel the sine at 3000 Hz, and the result will be zero.

4. **20 points** A certain device is controlled via a keyboard that has only the alphabetic keys, A through Z. Assume that you can only press one key at a time (unlike a real computer keyboard). A portion of the specification this device says:

> To shut down the device, the user should hit keys “A”, “B”, and “C” in sequence, one at a time.

The designer of the device uses a state machine to get the desired behavior, and defines the state machine with the following table:

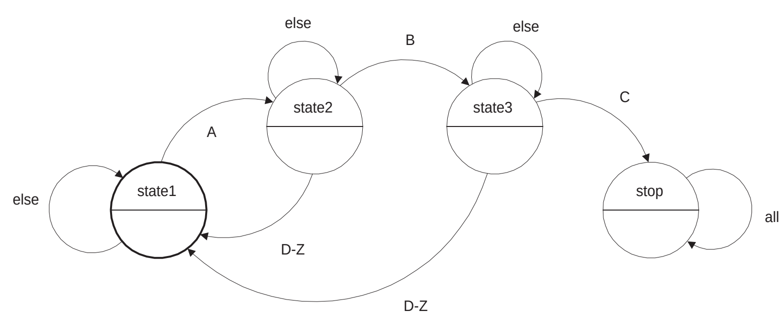
state	*next state under input*			
	A	B	C	D-Z
state1	state2			
state2		state3		state1
state3			stop	state1
stop				

The start state is *state1*. The blank entries indicate that the state machine stays in the same state. There is no output from this state machine. Notice that the response to the whole set of keys D through Z is given in a single column of the table, for compactness.

(a) Sketch and label the state transition diagram corresponding to the above table. Be sure your diagram specifies all aspects of the behavior. In particular, show “else” transitions (if any) explicitly.

(b) The table has a bug in it. Its behavior does not quite match the specification. Identify the bug; in particular, give a sequence of inputs that illustrates the bug, and give a state transition table that corrects the bug.

Solution:

(a) 

(b) The sequence [A, B, B, C] will cause the machine to enter the stop state, contrary to the specification. The state transition table is fixed below:

state	*next state under input*			
	A	B	C	D-Z
state1	state2			
state2		state3	state1	state1
state3	state2	state1	stop	state1
stop				