THE MANY FACES OF FREQUENCY 6

Electrical Engineering 20N
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1 The Minutiae of Continuous Time

We live in *continuous time*: to us, time is a continuum, each moment connected seamlessly to the next. Signals that live in continuous time are called, wait for it, **continuous-time signals**. The sound of your alarm clock blaring or of your lab TA droning on are examples of continuous-time sound signals. To be more mathematically precise, a continuous-time signal f is a function that maps a real number, usually a specific point in time t, to either a real number (if it is an *analog* signal) or an integer (if it is a *digital* signal). We measure continuous time using conventional units of time: t can be measured in terms of either seconds or minutes or millenia. In this course, t will usually be measured in *seconds*.

CONTINUOUS-TIME SIGNAL

One of the simplest, yet most ubiquitous, continuous-time signals is the *cosine wave*. You have probably seen this signal before in many other contexts. It plays a pivotal role in the study of signals because, as you will see in the last part of the course, *every* continuous-time signal can be decomposed as the sum of scaled and shifted cosine waves. *All* of them. Yes, even that one right behind you.[1]

In general, a cosine wave is represented by the equation

$$f(t) = A\cos(\omega_0 t + \phi),$$

where $f(t)$ is the value of the cosine wave at any one point, A is the **amplitude**, ϕ is the **initial phase** (the "shift") of the cosine wave, and ω_0 is its **angular frequency**.

AMPLITUDE
INITIAL PHASE
ANGULAR FREQUENCY

The word "frequency" is a loaded term: you will see it used in many different ways in this course. Nonetheless, in all of these uses, it has essentially the same meaning: it measures how often a signal changes in a given time period, or how often something new happens to a signal. Here, for example, ω_0 measures how many oscillations occur in a given second, measured in *radians per second*.

However, this unit of *radians per second* is somewhat unnatural and unintuitive. In many real-life applications, we prefer to measure the frequency of a continuous-time signal by counting the number of cycles the signal completes in a given second. A periodic signal starts off at a specific value and completes a **cycle** when it returns to that value. Note that the $\cos$ function is periodic with a period of 2π: this implies that

CYCLE

[1] Ha, made you look.

a cycle is equivalent to 2π radians. If we count the number of cycles the cosine wave completes in one second—call this value f_0—and multiply it by 2π, the result should be ω_0. To summarize,

$$\omega_0 = 2\pi f_0,$$

and so

$$f(t) = A\cos(\omega_0 t + \phi) = A\cos(2\pi f_0 t + \phi).$$

f_0, as you may guess, is measured in *cycles per second*, which is otherwise known as *Hertz* (*Hz*). This is the quantity that we use to measure sounds in. For example, musicians define the note A to be the sinusoidal sound wave with frequency 440 Hz; bats interact with the world using sounds at frequencies in the range 20kHz to 100kHz; and the human ear can detect sounds in the frequency range from 20Hz to 20kHz. Both ω_0 and f_0 are described as the **continuous-time frequency** of the signal, and context and units determine which specific quantity is being referred to.

CONTINUOUS-TIME FREQUENCY

For simplicity, for the rest of the document, we will refer to our exemplar continuous-time cosine wave as $f_{\mathsf{C}}(t)$ with no initial phase. Then,

$$f_{\mathsf{C}}(t) = A\cos(\omega_0 t).$$

2 Sampling

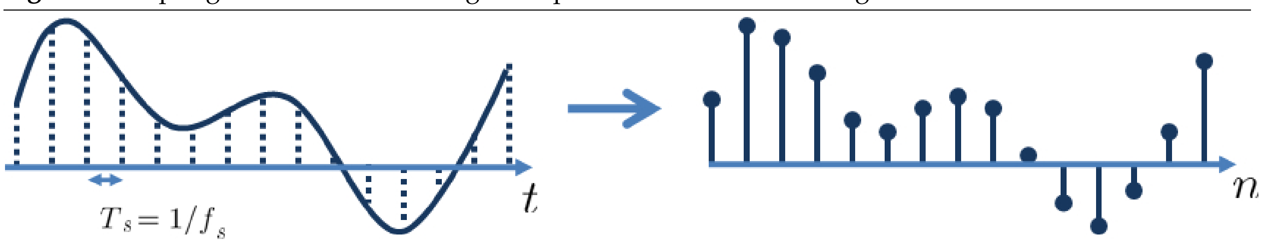
Matters become interesting when we attempt to store continuous-time signals on our computers. This attempt has many motivations: perhaps the most relevant is that we want to store a signal on our computers, where we can easily work on the signal. The manipulation of signals using computers and computer-based devices is at the crux of digital signal processing. However, we have one major issue: continuous-time signals have an uncountably infinite set of values, while computers can only store a finite set of values: there are far, far more "time points" in the period from now until the next second than can ever be stored on any computer.

We do the next best thing: we **sample** the continuous-time signal to produce the corresponding **discrete-time signal**, as shown in Figure 1.

SAMPLE

DISCRETE-TIME SIGNAL

Figure 1 Sampling a continuous-time signal to produce a discrete-time signal.



As the figure implies, when we sample a continuous-time signal, we record the value of the continuous-time signal at equally spaced time points. Each such recorded value is called a **sample**, a snapshot of the continuous-time signal at a specific time point. The resultant signal is called a **discrete-time signal** because time has been "discretized", or broken down into integer pieces. In continuous-time, we measured time in *seconds*; in discrete-time, we measure 'time' in *samples*. You can consider each sample as a position in an array, and indeed this is often how discrete-time signals are stored in a computer.

SAMPLE

We define **sampling period**, T_s, as the time period between two successive samples. Since we collect samples in continuous-time, T_s is measured in *seconds per sample*. The inverse of sampling period is termed **sampling frequency** (there's that word again). The sampling frequency, f_s, is measured in units of *samples*

SAMPLING PERIOD

SAMPLING FREQUENCY

per second: in other words, how many samples of the continuous-time signal are we collecting per second?

There are, of course, more factors that we have to consider when sampling a continuous-time signal: for example, what sampling frequency should we use to completely capture all the necessary information from a signal? There is a whole body of research and study devoted to these questions, under the umbrella of *sampling theory*, which we will explore in EE120. For now, however, these questions are beyond the scope of EE20. All we will need to consider is the following: the derived discrete-time signal is a completely distinct signal from its original continuous-time version. Thus, it has distinct properties: specifically, it has its own set of **discrete-time frequencies** that may be different from the continuous-time frequencies of the original signal. How do we relate the two?

DISCRETE-TIME FREQUENCIES

3 Moving from Discrete-Time to Continuous-Time

Let us bring back the continuous-time cosine wave, $f_{\mathsf{C}}(t)$. We sample it with a sampling frequency of f_s to generate its discrete-time counterpart, $f_{\mathsf{D}}(n)$. We know that

$$f_{\mathsf{C}}(t) = A\cos(\omega_0 t).$$

We also note that the nth sample of the discrete-time signal has the value of the continuous-time signal at time nT_s. One way to see this is to think of each sample as a fencepost in a picket fence: if the fenceposts are separated by a distance of T_s, at what distance is the nth fencepost positioned?

To summarize, we have

$$f_{\mathsf{D}}(n) = f_{\mathsf{C}}(nT_s) = A\cos(\omega_0 \cdot (nT_s)).$$

Assuming (and this is a huge assumption) that our resulting discrete-time signal is periodic, we should be able to represent the signal as

$$f_{\mathsf{D}}(n) = A\cos(\Omega n),$$

where Ω is the **discrete-time frequency** of the signal, measured in *radians per sample* (why?). This is analogous to how we represented the periodic continuous-time cosine wave as

$$f_{\mathsf{C}}(t) = A\cos(\omega_0 t),$$

where ω_0 was the **continuous-time frequency** of the signal.

We are close to our final relationship. Since

$$f_{\mathsf{D}}(n) = A\cos(\Omega n) = A\cos\left((\omega_0 T_s)\cdot n\right),$$

we conclude that

$$\boxed{\Omega = \omega_0 T_s = \frac{\omega_0}{f_s} = \frac{2\pi f_0}{f_s}},$$

where f_0 and ω_0 are the continuous-time frequencies of the original continuous-time signal, measured in *Hertz* and *radians per second* respectively, f_s is the sampling frequency measured in *samples per second*, and Ω is the discrete-time frequency of the derived discrete-time signal, measured in *radians per sample*. Ta-da! This is an important relationship that relates continuous-time signals and their discrete-time sampled counterparts.