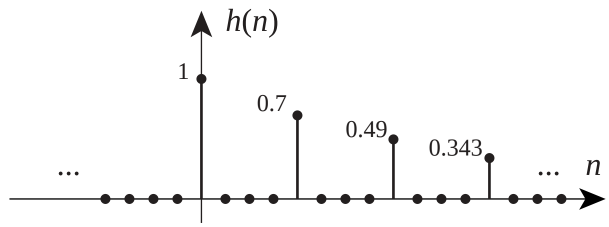

Figure 5.3: Impulse response of the echo example 5.16 for $\alpha = 0.7$ and $N = 4$.

Many interesting systems, unlike this example, have an **infinite impulse response**, and are referred to as **IIR systems**.

Example 5.16: Recall from example 5.6 that an echo effect can be obtained for audio signals by realizing the following difference equation,

$$\forall\, n \in Integers, \quad y(n) = x(n) + \alpha y(n-N).$$

The impulse response h can be obtained by simply letting the input be an impulse, $x = \delta$, and finding the output $y = h$. That is

$$\forall\, n \in Integers, \quad h(n) = \delta(n) + \alpha h(n-N). \tag{5.39}$$

But of course, this means that

$$\forall\, n \in Integers, \quad h(n-N) = \delta(n-N) + \alpha h(n-2N).$$

Substituting this back into (5.39), we get

$$\forall\, n \in Integers, \quad h(n) = \delta(n) + \alpha\delta(n-N) + \alpha^2 h(n-2N).$$

But of course,

$$\forall\, n \in Integers, \quad h(n-2N) = \delta(n-2N) + \alpha h(n-3N),$$

so

$$\forall\, n \in Integers, \quad h(n) = \delta(n) + \alpha\delta(n-N) + \alpha^2\delta(n-2N) - \alpha^3 h(n-3N).$$

Continuing in this fashion, we see that the impulse response of the echo system is the original impulse and an infinite set of echos (delayed and scaled impulses). This can be written compactly as follows,

$$\forall\, n \in Integers, \quad h(n) = \sum_{k=0}^{\infty} \alpha^k \delta(n-kN).$$

This impulse response is plotted in figure 5.3 for $N = 4$ and $\alpha = 0.7$. In that figure, you can see that the original impulse gets through the system at $n = 0$, while the first echo is scaled by 0.7 and delayed to $n = 4$, and the second echo is scaled by 0.7^2 and delayed to $n = 8$.

We can check this impulse response using the state-space representation of example 5.6 in (5.38). From example 5.6, we have that $d = 1$, so (5.38) is obviously correct for $n < 0$ and $n = 0$. Checking it for $n > 0$ is somewhat more involved. Assuming $N = 4$, we have from example 5.6

$$A = \begin{bmatrix} 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix}, \; d = \begin{bmatrix} 1 \end{bmatrix}.$$

To use (5.38) we need to know A^{n-1}. It is easy to check that

$$A^2 = \begin{bmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; A^3 = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; A^4 = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} = \alpha I,$$

where I is the 4×4 identity matrix. Thus,

$$A^5 = \alpha A, \; A^6 = \alpha A^2, \; A^7 = \alpha A^3, \; A^8 = \alpha A^4 = \alpha^2 I.$$

The pattern continues. Note that because of the particular structure of b, $A^{n-1}b$ is simply the first column of A^{n-1}. Thus,

$$h(1) = c^T A^0 b = [0\;0\;0\;\alpha] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

Similarly,

$$h(2) = c^T A b = [0\;0\;0\;\alpha] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0,$$

and

$$h(3) = c^T A^2 b = [0\;0\;0\;\alpha] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0.$$

Only when we get to $h(4)$ do we get a non-zero result,

$$h(4) = c^T A^3 b = [0\;0\;0\;\alpha] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \alpha.$$

Continuing in this fashion, we can determine that 3 of every four samples of the impulse response are zero, and the non-zero ones have the form $\alpha^{n/4}$ where n is a multiple of four, in perfect agreement with figure 5.3.

The echo system of the previous example is called an **infinite impulse response** (**IIR**) system because the response to an impulse never completely dies out. Here is another example of an IIR system.